

AFFINE FUNCTIONS ON COMPACT CONVEX SETS

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INTRODUCTION

This work deals with compact convex sets and affine functions on them. The approach differs from that of Alfsen, in his book "Compact Convex Sets and Boundary Integrals", Springer-Verlag, Berlin-Heidelberg-New York, 1971, in that the measure theoretic material has been kept to a minimum. Only results needed for the study of affine functions have been included, and some topics such as simplicial and central measures omitted altogether. There is some overlap in content, especially in the earlier part of the work, but often proofs are different.

The references contain only those papers directly related to material here which either contain more than is included here, or else which are not referred to in Alfsen's book. That should be consulted for a more complete bibliography.

The references applicable to each chapter are as follows:

Chapter I - [12], [15] & [25].

Chapter II - [1], [8], [9], [10], [11], [13], [16], [19], [23], [28], [32], [33], [38] & [43].

Chapter III - [1], [3], [6], [7], [15], [17], [18], [26], [29], [30], [31] & [40].

Chapter IV - [4], [5], [24], [37] & [41].

Chapter V - [2], [22], [27], [28], [38] & [42].

Chapter VI - [7], [11], [14], [29] & [34].

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PREREQUISITES

The following is a brief summary of the results from various branches of mathematics that we shall assume.

Topology.

Apart from the basic properties of metric and topological spaces, we shall need the definitions of Baire spaces and of compact Hausdorff spaces. We shall also apply Urysohn's lemma on compact Hausdorff spaces.

All that we require may be found in "General Topology" by J.L. Kelley, Van Nostrand, Princeton, 1955.

Measure Theory.

Two results from abstract measure theory that we will need are the dominated convergence theorem of Lebesgue and the Radon-Nikodym theorem. We require these for finite measures only. We shall need the definitions and basic properties of Baire and regular Borel measures on compact Hausdorff spaces, and the Riesz representation theorem for such spaces.

All that we require may be found in "Measure Theory" by P.R. Halmos, Van Nostrand, Princeton, 1950.

Function Spaces.

We employ the well-known theorems of Ascoli, Dini and Stone-Weierstrass for the space of continuous real valued functions on a compact Hausdorff space.

All that we require may be found in "Linear Operators I" by N. Dunford and J.T. Schwartz, Interscience, New York, 1958.

Functional Analysis.

For locally convex spaces we shall assume knowledge of

the Hahn-Banach theorem (both in its extension and separation versions), the Krein-Milman theorem and Milman's converse theorem. In Banach spaces, apart from elementary duality theory, definition of both bounded and compact operators and familiarity with the weak* topology, we make frequent use of Alaoglu's theorem, the Open Mapping theorem and the Krein-Smulian theorem.

All that we require may be found in "Topological Vector Spaces I" by G. Kothe, Springer-Verlag, Berlin-Heidelberg-New York, 1969.

Banach Algebras.

Familiarity with the definition of a Banach algebra and of an algebra homomorphism might be useful in places. No depth of knowledge is assumed.

Ordered Vector Spaces.

The material we need on ordered vector spaces may not be so familiar, so we shall summarise all that we require here.

Let E be a real vector space. A set $P \subseteq E$ is a wedge if $\lambda P \subseteq P$ for all $\lambda \in \mathbb{R}^+$ and if $P + P \subseteq P$. If also $P \cap (-P) = \{0\}$ P is a cone. E is positively generated if $E = P - P$. If P is a wedge in E define $x \gg y$ to mean $x - y \in P$. Then " $\gg$ " is a partial order on E such that

$$(1) \quad x \gg y \text{ implies } x + z \gg y + z \text{ for all } z \in E.$$

$$(2) \quad x \gg y, \lambda \gg 0 \text{ implies } \lambda x \gg \lambda y.$$

Conversely if " $\gg$ " is a partial order on E satisfying (1) and (2) then $P = \{x \in E : x \gg 0\}$ is a wedge in E such that $x \gg y$ if and only if $x - y \in P$. In this case P is a cone if and only if the order in E is proper; i.e. if $x \gg y$ and $y \gg x$ then $x = y$. By an ordered vector space we mean a pair (E, P) where P is a cone in E ,

but we usually just write E , and assume that this has the order derived from P as above. We use " $x \succcurlyeq y$ " and " $y \preccurlyeq x$ " interchangeably. The order on E is Archimedean if $x \preccurlyeq ny$ for all $n \in \mathbb{N}^+$ implies $x \preccurlyeq 0$.

An ordered vector space E is a vector lattice if for all $x, y \in E$ the pair (x, y) has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$ in E . I.e. (for example) $x \vee y \succcurlyeq x, y$ and $z \succcurlyeq x, y$ implies $z \succcurlyeq x \vee y$. We use the notation $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$. For E to be a vector lattice it is sufficient, for example, that x^+ exist for each $x \in E$. A vector lattice E is complete if whenever $\{x_\alpha : \alpha \in A\} \subset E$, $y \in E$ and $x_\alpha \preccurlyeq y$ for all $\alpha \in A$ there is $x \in E$ with $x_\alpha \preccurlyeq x$ for all $\alpha \in A$ and $x_\alpha \preccurlyeq z$ for all $\alpha \in A$ implies that $x \preccurlyeq z$ (a similar statement with inequalities reversed then holds automatically).

If E and F are ordered vector spaces a linear operator $T : E \rightarrow F$ is positive if $x \succcurlyeq 0$ implies $Tx \succcurlyeq 0$. If E and F are vector lattices we call the linear operator $T : E \rightarrow F$ a lattice homomorphism if for each $x, y \in E$, $T(x \vee y) = (Tx) \vee (Ty)$. A lattice homomorphism must be positive.

If E is an ordered vector space a vector subspace $F \subset E$ is cofinal if for each $x \in E$ there is $y \in F$ with $x \preccurlyeq y$. A version of the Hahn-Banach theorem known as the monotone extension theorem is of occasional use to us. This states that if E is an ordered vector space, G a complete vector lattice and F a cofinal subspace of E then every positive linear operator from F into G has a positive linear extension from E into G .

All that we require may be found in "Ordered Linear Spaces" by G.J.O. Jameson, Springer Lecture Notes in Mathematics, 1970. Familiarity with more would be useful, though explicit knowledge of more is not assumed.

The following notation will be used throughout with no further explanation. All other definitions and conventions will be defined in the body of the text, and are also indexed at the end.

$\text{co}(A)$ - convex hull of A .

$\overline{\text{co}}(A)$ - closed convex hull of A .

E_1 - closed unit ball of the normed space E .

$\|f\|_A$ - $\sup\{|f(a)| : a \in A\}$ where f is a bounded function on X and $A \subset X$.

$\mathbb{R}$ - the real numbers.

$\mathbb{R}^+$ - the non-negative reals.

$\mathbb{Z}$ - the integers.

$\mathbb{Z}^+$ - the non-negative integers.

$\text{supp}(\mu)$ - support of the Radon measure μ .

CHAPTER I

COMPACT CONVEX SETS

1 Representation Theorems.

If C is a convex set in a vector space E , a subset F of C is a face if it is convex and if $x, y \in C$, $0 < \lambda < 1$ and $\lambda x + (1 - \lambda)y \in F$ together imply $x, y \in F$. A point $x \in C$ such that $\{x\}$ is a face of C is termed an extreme point of C . $\partial_e C$ denotes the set of all extreme points of C . Trivially we have:

LEMMA I.1.1. If C is a convex set, and F a face of C , then every face of F is a face of C .

COROLLARY I.1.2. If F is a face of C then $\partial_e F \subset \partial_e C$.

A function $f : C_1 \rightarrow C_2$ (where C_1, C_2 are convex sets) is affine if $x, y \in C_1$ and $0 < \lambda < 1$ imply that $f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y)$.

LEMMA I.1.3. If $f : C_1 \rightarrow C_2$ is affine and onto, and F is a non-empty face of C_2 , then $f^{-1}(F)$ is a non-empty face of C_1 .

If $x, y \in f^{-1}(F)$ and $0 < \lambda < 1$, then $f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y) \in F$ as F is convex. I.e. $f^{-1}(F)$ is convex. If, now, $x, y \in C_1$, $0 < \lambda < 1$ and $\lambda x + (1 - \lambda)y \in f^{-1}(F)$, then $f(\lambda x + (1 - \lambda)y) \in F$. Hence $\lambda f(x) + (1 - \lambda)f(y) \in F$, so $f(x), f(y) \in F$, since F is a face of C_2 . I.e. $x, y \in f^{-1}(F)$, so $f^{-1}(F)$ is a face. That $f^{-1}(F)$ is non-empty is because F is non-empty and f is onto.

By a compact convex set, we mean a compact convex non-empty subset of a real locally convex space. If K is a compact convex set then the Krein-Milman theorem tells us

that $\partial_e K \neq \emptyset$, and that $K = \overline{\text{co}}(\partial_e K)$, the closed convex hull of its extreme points. We denote by $A(K)$ the vector space of all continuous affine real-valued functions on K . With the supremum norm $A(K)$ is a Banach space. We order $A(K)$ by the cone $A(K)^+$ of all functions $a \in A(K)$ with $a(k) \geq 0$ for all $k \in K$. If $f \in E^*$, the topological dual of a locally convex space E , and K is a compact convex subset of E , then $f|_K \in A(K)$. Thus we may invoke the Hahn-Banach theorem to assert that the elements of $A(K)$ separate the points of K . We can use this fact to say a little more about $\partial_e K$:

THEOREM I.1.4. For the relative topology induced by K , $\partial_e K$ is a Baire space.

If $a \in A(K)$, let $U_{a,\alpha} = \{k \in K : a(k) < \alpha\}$ and $F_{a,\alpha} = \{k \in K : a(k) \leq \alpha\}$. If $k \in \partial_e K$ we claim that the family $\{F_{a,\alpha} : k \in U_{a,\alpha}\}$ is a neighbourhood base for k in K . The fact that the functions in $A(K)$ separate the points of K implies that the intersection of such $F_{a,\alpha}$ is $\{k\}$. As the sets $F_{a,\alpha}$ are compact, it suffices to show that the family is downward filtering. Suppose $k \in U_{a,\alpha} \cap U_{b,\beta}$, then $k \notin K \setminus U_{a,\alpha}$ and $k \notin K \setminus U_{b,\beta}$. These last two sets are compact and convex, hence so is their convex hull, C . By Milman's theorem $k \notin C$, for if it were it would certainly be extreme in C , but $\partial_e C \subset (K \setminus U_{a,\alpha}) \cup (K \setminus U_{b,\beta})$, so this cannot be. We may now use Hahn-Banach separation to obtain $f \in E^*$ with $f(k) < \gamma$, $f(x) > \gamma$ if $x \in C$. Now take $c = f|_K$, then $F_{c,\gamma} \subset U_{a,\alpha} \cap U_{b,\beta}$ and $k \in U_{c,\gamma}$ as required.

Let $\{O_n\}_{n=1}^{\infty}$ be a collection of open dense subsets of $\partial_e K$. We must show that if V is open in $\partial_e K$ and non-empty, then

$V \cap \bigcap_{n=1}^{\infty} O_n \neq \emptyset$. We may clearly suppose that the family $\{O_n\}$ is decreasing. Suppose $O_n = U_n \cap \partial_e K$ and $V = U_{a_1, \alpha_1} \cap \partial_e K$ where $\{U_n\}$ is a decreasing family.

We define a sequence of pairs (a_i, α_i) as follows. We already have (a_1, α_1) ; suppose we already have defined (a_n, α_n) . Let $x \in \partial_e K \cap U_{a_n, \alpha_n} \cap U_{n+1}$. Then $U_{a_n, \alpha_n} \cap U_{n+1}$ is a neighbourhood of x in K , so there exist (a_{n+1}, α_{n+1}) with $x \in U_{a_{n+1}, \alpha_{n+1}} \subset F_{a_{n+1}, \alpha_{n+1}} \subset U_{a_n, \alpha_n} \cap U_{n+1}$. We choose this (a_{n+1}, α_{n+1}) for the next term.

The family F_{a_n, α_n} is a decreasing family of non-empty compact sets, so has a non-empty intersection F . $F \subset U_{a_1, \alpha_1}$ and $F \subset \bigcap_{n=1}^{\infty} U_n$. It is easily verified that F is compact and convex and that $K \setminus F$ is also convex. It is now clear that $\partial_e F \subset \partial_e K$. If $k \in \partial_e F$, then $k \in \partial_e K \cap U_n = O_n$ and $k \in \partial_e K \cap U_{a_1, \alpha_1} = V$. The Krein-Milman theorem now asserts that the desired intersection is non-empty.

Suppose Ω is a compact Hausdorff space, and A is a closed subspace of $C(\Omega)$ that contains the constant functions and separates the points of Ω . The state space of A , $S(A)$, is the weak* compact convex subset of A^* , $\{f \in A^* : f(1_\Omega) = \|f\| = 1\}$ (where 1_Ω is the function on Ω that is 1 everywhere).

If A is all of $C(\Omega)$, then $S(A)$ may be identified with $P(\Omega)$, the set of regular Borel measures on Ω that are positive and with total mass 1, i.e. the probability measures on Ω . This set is of considerable interest in what is to follow.

If $\mu \in P(\Omega)$ and D is a Borel subset of Ω , we define a measure μ_D on the Borel sets of Ω by $\mu_D(B) = \mu(D \cap B)$. It is easily verified that μ_D is a regular measure, and that if $\mu(D) \neq 0$ then $\mu(D)^{-1} \mu_D \in P(\Omega)$. If $\omega \in \Omega$, we denote by ε_ω the

point mass at w , i.e. $\xi_w(B) = 1$ if $w \in B$,
 $= 0$ if $w \notin B$.

PROPOSITION I.1.5. If $\mathcal{N}$ is a compact Hausdorff space then
 $\partial_e(P(\mathcal{N})) = \{\xi_w : w \in \mathcal{N}\}$.

That ξ_w is extreme is easily verified. If $\mu, \nu \in P(\mathcal{N})$
and $\xi_w = \alpha\mu + (1 - \alpha)\nu$ with $0 < \alpha < 1$, then

$$1 = \xi_w(\{w\}) = \alpha\mu(\{w\}) + (1 - \alpha)\nu(\{w\})$$

which is only possible if $\mu(\{w\}) = \nu(\{w\}) = 1$, since
 $\mu(\{w\}), \nu(\{w\}) \leq 1$. As μ and ν have total mass 1,
 $\mu = \nu = \xi_w$.

Suppose conversely that μ is extreme in $P(\mathcal{N})$. We show
that there is a closed subset D of $\mathcal{N}$ with $0 < \mu(D) < 1$ if μ
is not ξ_w for some $w \in \mathcal{N}$. Consider the collection of all
closed subsets of $\mathcal{N}$ of μ -measure 1. If D_1 and D_2 are two such,
then $\mu(D_1 \cap D_2) = 0$ or 1, else our claim is justified. If
 $\mu(D_1 \cap D_2) = 0$ then $\mu(D_1 \setminus D_2) = \mu(D_2 \setminus D_1) = 1$, so that
 $\mu(\mathcal{N}) \geq 2$ - which is impossible. Thus the family has the
finite intersection property, so we may choose w in the
intersection. If $\mu(\{w\})$ is non-zero we either have the
desired closed set (if $\mu(\{w\}) < 1$) or else have proved the
whole result (if $\mu(\{w\}) = 1$). Thus we may find, by regularity,
an open set containing w of μ -measure less than $1/2$. The
complement of this is either a closed set with measure between
0 and 1, or else has measure 1, contradicting the definition of
 w . In either case we have the desired set D .

Now $\mu \neq \mu(D)^{-1}\mu_D$ and $\mu = \mu(D)(\mu(D)^{-1}\mu_D) +$
 $(1 - \mu(D))(\mu(\mathcal{N} \setminus D)^{-1}\mu_{\mathcal{N} \setminus D})$, contradicting the extremality
of μ , and the result is proved.

Suppose $A \subset B \subset C(\mathcal{N})$ where A and B are closed subspaces containing the constants and separating the points of $\mathcal{N}$. If $f \in S(B)$, then $f|_A \in S(A)$. The Hahn-Banach theorem allows us to assert that this restriction map is onto. The definition of the weak* topology makes it clear that this restriction map is continuous, and it is clearly affine. We have thus proved:

LEMMA I.1.6. The restriction map from $S(A)$ to $S(B)$ is continuous, affine and onto.

COROLLARY I.1.7. $\partial_e(S(A)) \subset \{\varepsilon_\omega|_A : \omega \in \mathcal{N}\}$.

If $f \in \partial_e(S(A))$, by lemma I.1.3 $\{\mu \in P(\mathcal{N}) : \mu|_A = f\}$ is a closed face of $P(\mathcal{N})$, and is also non-empty (here $\mu|_A$ means that linear functional on A which assigns to each function its integral with respect to μ). Applying the Krein-Milman theorem, and noting that the functions in A separate the points of $\mathcal{N}$, and hence the extreme points of $P(\mathcal{N})$, we see that $\{\mu \in P(\mathcal{N}) : \mu|_A = f\} = \{\varepsilon_\omega\}$ for some $\omega \in \mathcal{N}$. The result is now clear.

In particular if we take K to be a compact convex set, $A(K)$ is a subspace of $C(K)$ of the required form. We can form $S(A(K))$, and the restriction map takes $P(K)$ onto $S(A(K))$. If $k \in K$ then $\tilde{k}$, the evaluation at k , belongs to $S(A(K))$. It is obviously of interest to know if all of $S(A(K))$ arises in this way. In fact we have:

THEOREM I.1.8. The map $k \mapsto \tilde{k}$ is an affine homeomorphism of K onto $S(A(K))$.

The map is obviously affine, and is (1,1) as the functions in $A(K)$ separate the points of K . Continuity is

because if $f \in A(K)^*$ and $\varepsilon > 0$, then $\{k : |\tilde{k}(a) - f(a)| < \varepsilon\} = \{k : |a(k) - f(a)| < \varepsilon\}$ is open by the continuity of a . We must finally show that the map is onto. As $\{\tilde{k} : k \in K\}$ is a compact convex subset of $S(A(K))$, it will suffice to prove that the map is onto $\partial_e S(A(K))$.

Let $\rho : P(K) \rightarrow S(A(K))$ be the restriction map, and $f \in \partial_e S(A(K))$. By I.1.3 and I.1.6 $\rho^{-1}(f)$ is a closed face of $P(K)$. By the Krein-Milman theorem this set has extreme points, which must be extreme points of $P(K)$. By I.1.5 these are of the form ξ_k for some $k \in K$. I.e. we have found $k \in K$ with $a(k) = f(a)$ for all $a \in A(K)$. Thus $k \mapsto \tilde{k}$ is onto.

COROLLARY I.1.9. If $\mu \in P(K)$ there is a unique point $k \in K$ with $a(k) = \int a d\mu$ for all $a \in A(K)$.

Combine the restriction to $A(K)$ with the inverse of the embedding of K into $S(A(K))$ to obtain existence. Uniqueness is because the functions in $A(K)$ separate the points of K .

This point is termed the resultant of μ , and is written $r(\mu)$. This notion allows us to give another characterisation of extreme points of K .

COROLLARY I.1.10. If $k \in K$ then k is extreme if and only if $r(\mu) = k$ implies $\mu = \xi_k$.

If k is not extreme, then $k = \lambda x + (1 - \lambda)y$ with $x, y \in K$, $0 < \lambda < 1$ and $x \neq y$. It is clear that $\lambda \xi_x + (1 - \lambda) \xi_y \neq \xi_k$, and that $r(\lambda \xi_x + (1 - \lambda) \xi_y) = k$. If conversely k is extreme, then $r^{-1}(k)$ is a closed non-empty face of $P(K)$. Any extreme point must be evaluation at $k' \in K$ by I.1.5. Clearly this must mean that $k = k'$. By the Krein-Milman theorem we may conclude that $r^{-1}(k) = \{\xi_k\}$.

COROLLARY I.1.11. Every bounded linear functional f on $A(K)$ is of the form $a \mapsto \alpha a(k_1) - \beta a(k_2)$ with $\alpha, \beta \geq 0$, $k_1, k_2 \in K$ and $\|f\| = \alpha + \beta$.

Extend f to a bounded linear functional on $C(K)$ by the Hahn-Banach theorem, with preservation of norm. I.e. we have $\mu \in M(K)$, the set of regular Borel measures on K , with $\mu(a) = f(a)$ and $\mu^+(K) + \mu^-(K) = \|f\|$. Put $\mu^+ = \alpha \nu$, $\mu^- = \beta \lambda$ with $\alpha, \beta \geq 0$ and $\nu, \lambda \in P(K)$, then $\|f\| = \alpha + \beta$. Take $k_1 = r(\nu)$ and $k_2 = r(\lambda)$ and the result is clear.

COROLLARY I.1.12. Let K be a compact convex subset of the locally convex space E . Any subspace of $A(K)$ which contains the constants and separates the points of K is dense in $A(K)$. In particular the functions of the form $k \mapsto f(k) + m$ for $f \in E^*$, $m \in \mathbb{R}$ form a dense subspace of $A(K)$.

Let H be such a subspace, if it is not dense there is $g \in A(K)^*$ with $g|_H = 0$, $g \neq 0$. By I.1.11 there are $\alpha, \beta \geq 0$ and $k_1, k_2 \in K$ with $g(a) = \alpha a(k_1) - \beta a(k_2)$ whenever $a \in A(K)$. Consider the constant function $1_K \in H$, then we see that $\alpha = \beta$. Choosing $h \in H$ with $h(k_1) \neq h(k_2)$ we see that $\alpha = \beta = 0$ so that $g = 0$. This completes the proof.

We shall henceforth consider K to be embedded in $A(K)^*$ in this manner without further comment. We introduce now a notion that will be of use to us later. If E is an ordered vector space, a convex set C is a base for the positive cone in E if any $x \in E^+$ can be written uniquely as λc for $\lambda \in \mathbb{R}^+$, $c \in C$. If E is lattice ordered, C is termed a simplex. It is easily verified that this definition does not depend on the particular choice of E . In particular if K is a compact convex set, K is a simplex if and only if $A(K)^*$ is a lattice. Recall

also that a partially ordered vector space has the Riesz decomposition property (R.D.P.) if whenever $x, y, z \geq 0$ and $0 \leq z \leq x + y$ there are z_1, z_2 with $z_1 + z_2 = z$ and $0 \leq z_1 \leq x, 0 \leq z_2 \leq y$. Equivalently the space has the Riesz separation property, i.e. if $x_1, x_2 \leq y_1, y_2$ then there is z with $x_1, x_2 \leq z \leq y_1, y_2$. This we will usually abbreviate to R.S.P.

LEMMA I.1.13. If $A(K)$ has the R.D.P. then K is a simplex.

It will suffice to prove that if $f \in A(K)^*$ then $f \vee 0$ exists in $A(K)^*$. Consider the map $h : a \mapsto \sup \{f(b) : 0 \leq b \leq a\}$ on $A(K)^+$. It is clear that $h(a) \geq f(a), 0$. If $a \in A(K)^+$ and $0 \leq b \leq a$ then $f(b) \leq \|f\| \|b\| \leq \|f\| \|a\|$. Thus $|h(a)| \leq \|f\| \|a\|$, so that h is a real-valued function.

h is additive on $A(K)^+$, for

$$\begin{aligned} h(a_1 + a_2) &= \sup \{f(b) : 0 \leq b \leq a_1 + a_2\} \\ &= \sup \{f(b_1 + b_2) : 0 \leq b_1 \leq a_1, 0 \leq b_2 \leq a_2\} \\ &= \sup \{f(b_1) : 0 \leq b_1 \leq a_1\} + \\ &\quad \sup \{f(b_2) : 0 \leq b_2 \leq a_2\} \\ &= h(a_1) + h(a_2). \end{aligned}$$

h may thus be extended to a linear functional on $A(K)$. Clearly

if $g \geq f, 0$ then $g(b) \geq f(b)$ whenever $b \geq 0$, so if $a \geq 0$ then $g(a) = \sup \{g(b) : 0 \leq b \leq a\} \geq \sup \{f(b) : 0 \leq b \leq a\} = h(a)$.

Thus it remains only to prove that h is bounded.

If $a \in A(K)$ then $a = \|a\| 1_K - (\|a\| 1_K - a)$ where $\|a\| 1_K \geq 0$ and $\|a\| 1_K - a \geq 0$. Thus

$$\begin{aligned} |h(a)| &\leq h(\|a\| 1_K) + h(\|a\| 1_K - a) \\ &\leq \|f\| (\|a\| + \|(\|a\| 1_K - a)\|) \\ &\leq 3 \|f\| \|a\|. \end{aligned}$$

I.e. $\|h\| \leq 3 \|f\|$ and the proof is complete.

We shall see later that the converse of this is also true.

We recall that an ideal in an ordered vector space is a linear subspace J such that $0 \leq x \leq y$ and $y \in J$ imply $x \in J$, and such that every element of J can be written as the difference of two positive elements of J . We can describe the ideals in $A(K)^*$. If $F \subset S(A(K))$ let $\tilde{F} = \{ \lambda f : \lambda \geq 0, f \in F \}$.

PROPOSITION I.1.14. The map $J \mapsto J \cap S(A(K))$ is a bijection from the ideals in $A(K)^*$ onto the faces of $S(A(K))$. The inverse is $F \mapsto \tilde{F} - \tilde{F}$.

Suppose J is an ideal, $x, y \in S(A(K))$ and $0 < \lambda < 1$. Suppose also that $\lambda x + (1 - \lambda)y \in J$. Then $0 \leq \lambda x \leq (1 - \lambda)y + \lambda x$ so $\lambda x \in J$. As J is a linear space $x \in J$ and thus $x \in J \cap S(A(K))$. Similarly y belongs to this set, so it is a face.

Now suppose that F is a face of $S(A(K))$. It is only necessary to verify that $0 \leq x \leq y \in \tilde{F}$ implies $x \in \tilde{F}$, for the other condition is obvious. Without loss of generality we may suppose $y \in F$ and $0 \neq x \neq y$. Then $x' = \lambda^{-1}x \in S(A(K))$ for some λ with $0 < \lambda < 1$, so $y = \lambda x' + (1 - \lambda)\frac{(y - x)}{1 - \lambda}$ where also $x', \frac{y - x}{1 - \lambda} \in S(A(K))$. Hence, as F is a face, $x', \frac{y - x}{1 - \lambda} \in F$. Thus $x = \lambda x' \in \tilde{F}$ as claimed. As $\tilde{F} \cap S(A(K)) = F$ the proof is complete.

COROLLARY I.1.15. If K is a simplex and F is a face of K , then F is a simplex.

This is because in a lattice our notion of ideal coincides with the usual notion of lattice ideal. Thus $\tilde{F} - \tilde{F}$ will be a lattice and F is a base for this, so F is a simplex.

Of some importance is the next proposition. We state part of the proof as a lemma since we shall use it a number of times later.

LEMMA I.1.16. Let W be any cone in a lattice ordered vector space E . If $u, w \in W \implies u \vee w \in W$ then $W - W$ is a sublattice of E .

Suppose $e_1 = u_1 - w_1, e_2 = u_2 - w_2$ with $u_i, w_i \in W$, then $e'_i = e_i + (w_1 + w_2) \in W$ for $i = 1, 2$. Thus

$$\begin{aligned} e_1 \vee e_2 &= (e'_1 - (w_1 + w_2)) \vee (e'_2 - (w_1 + w_2)) \\ &= e'_1 \vee e'_2 - (w_1 + w_2) \in W - W. \end{aligned}$$

PROPOSITION I.1.17. If K is a compact convex set, the following are equivalent:

- (1) K is metrisable.
- (2) $C(K)$ is separable.
- (3) $A(K)$ is separable.

(1) $\implies$ (2). Let $\{U_n\}$ be a countable neighbourhood base for K . If $\bar{U}_n \cap \bar{U}_m = \emptyset$, we may use Urysohn's lemma plus compactness to obtain $f_{n,m} \in C(K)$ with $0 \leq f_{n,m} \leq 1$, $f_{n,m}|_{\bar{U}_n} = 1$ and $f_{n,m}|_{\bar{U}_m} = 0$. These functions clearly separate the points of K . We denote by $\mathfrak{F}$ the set $\{f_{n,m} : \bar{U}_n \cap \bar{U}_m = \emptyset\}$. Let V denote the cone generated by $\mathfrak{F} \cup \{1_K\}$, it clearly having a countable dense subset V' , the set of all sums of rational multiples of elements from $\mathfrak{F} \cup \{1_K\}$. Now let W (resp. W') be the set of all finite suprema of elements of V (resp. V'). It is clear that W' is a countable dense subset of W . W is a cone, for if $\bigvee_{i=1}^n l_i \in W$ and $\bigvee_{j=1}^m m_j \in W$ ($l_i, m_j \in V$), then $\lambda \geq 0$ implies $\lambda(\bigvee_{i=1}^n l_i) = \bigvee_{i=1}^n (\lambda l_i) \in W$. Similarly $(\bigvee_{i=1}^n l_i) + (\bigvee_{j=1}^m m_j) = \bigvee_{i=1}^n \bigvee_{j=1}^m (l_i + m_j) \in W$. By Lemma I.1.16 $W - W$ is a sublattice of $C(K)$, and is hence dense in $C(K)$ by an application of the Stone-Weierstrass theorem. As $W' - W'$ is dense in $W - W$ and is countable, it follows that $C(K)$ is separable.

(2) $\implies$ (3) is immediate.

(3) $\implies$ (1). Let $(a_n)_{n=1}^{\infty}$ be dense in $A(K)$. Define

$$\rho(k, k') = \sum_{n=1}^{\infty} 2^{-n} |a_n(k) - a_n(k')| / (1 + |a_n(k) - a_n(k')|),$$
which is a metric on K (the fact that ρ separates the points of K is because of the density of $(a_n)_{n=1}^{\infty}$ in $A(K)$). The topology induced by ρ is a Hausdorff topology on K which is weaker than the original compact topology on K , so coincides with it.

Later we shall want to deal with linear operators taking values in an $A(K)$ -space. In this context the following result will be of use:

THEOREM I.1.18. Let E be a Banach space and K a compact convex set. If π is a continuous affine function from K into E^* when the latter is equipped with the weak* topology, define $A(\pi) : E \longrightarrow A(K)$ by

$$(A(\pi)(e))(k) = e(\pi(k)) \quad (e \in E, k \in K).$$

Then $A : \pi \longmapsto A(\pi)$ is a bijection from the set of all such π onto the set of all bounded linear operators from E into $A(K)$. Further $\|A(\pi)\| = \sup \{ \|\pi(k)\| : k \in K \}$.

It is clear that $A(\pi)e$ lies in $A(K)$ for each $e \in E$, and that $A(\pi)$ is linear. Also it is easily verified that A is (1,1). To see that A is onto, suppose $T : E \longrightarrow A(K)$ is a bounded linear operator and let $\pi : K \longrightarrow E^*$ be the composition of $T^*|_{S(A(K))}$ with the natural injection of K onto $S(A(K))$ (T^* denotes the adjoint operator to T taking $A(K)^*$ into E^* , which is weak* continuous). It is easily verified that π is continuous, affine, and that $T = A(\pi)$.

Finally:

$$\begin{aligned}
 \|A(\pi)\| &= \sup \{ \|A(\pi)(e)\| : e \in E_1 \} \\
 &= \sup \{ \|(A(\pi)(e))(k)\| : e \in E_1, k \in A(K)^* \} \\
 &= \sup \{ \|(A(\pi)(e))(k)\| : e \in E_1, k \in S(A(K)) \} \\
 &= \sup \{ \|e(\pi k)\| : e \in E_1, k \in K \} \\
 &= \sup \{ \|\pi k\| : k \in K \}.
 \end{aligned}$$

We shall also need:

THEOREM I.1.19. With the notation of I.1.18, $A(\pi)$ is compact if and only if π is continuous for the norm topology on E^* .

By Ascoli's theorem $A(\pi)(E_1)$ is relatively compact if and only if this is an equicontinuous subset of $A(K)$ (the set necessarily being bounded). I.e. if $k_\gamma \rightarrow k$ in the given topology on K then $k_\gamma \rightarrow k$ uniformly on $A(\pi)(E_1)$. Thus if $k_\gamma \rightarrow k$ in the given topology for K , we have the following equivalences:

$$\begin{aligned}
 &\pi k_\gamma \rightarrow \pi k \text{ (in norm)} \\
 \iff &e(\pi k_\gamma) \rightarrow e(\pi k) \text{ (uniformly on } E_1) \\
 \iff &A(\pi)(e)(k_\gamma) \rightarrow A(\pi)(e)(k) \text{ (uniformly on } E_1) \\
 \iff &k_\gamma \rightarrow k \text{ (uniformly on } A(\pi)(E_1)).
 \end{aligned}$$

I.e. $A(\pi)(E_1)$ is relatively compact if and only if π is norm continuous.

Of particular interest is the case when E is of the form $A(K')$ for K' a compact convex set:

THEOREM I.1.20. If K and K' are compact convex sets, and $\pi : K' \rightarrow A(K)^*$ is weak* continuous and affine then the following are equivalent:

- (1) $A(\pi)$ satisfies $A(\pi)(1_K) = 1_K$, and either (a) $a \geq 0$ implies $A(\pi)a \geq 0$, or (b) $\|A(\pi)\| = 1$.
- (2) $\pi(K') \subset S(A(K))$.

$$\begin{aligned}
 (1a) \implies (1b), \text{ for } \|a\| \leq 1 &\implies -1_K \leq a \leq 1_K \\
 &\implies -1_{K'} \leq A(\pi)(a) \leq 1_{K'}, \\
 &\implies \|A(\pi)(a)\| \leq 1.
 \end{aligned}$$

(1b) $\implies$ (2). If $k' \in K'$, then

$$\begin{aligned}
 (\pi k')(1_K) &= A(\pi)(1_K)(k') \\
 &= (1_{K'})(k') = 1.
 \end{aligned}$$

So in particular $\|\pi k'\| \geq 1$. By (b)

$$\begin{aligned}
 \|\pi k'\| &= \sup \{ |a(\pi k')| : a \in A(K)_1 \} \\
 &= \sup \{ |(A(\pi)(a))(k')| : a \in A(K)_1 \} \\
 &\leq \sup \{ |f(k')| : f \in A(K')_1 \} \\
 &= 1.
 \end{aligned}$$

I.e. $\|\pi k'\| = 1$. Hence $\pi k' \in S(A(K))$.

That (2) $\implies$ (1a) and (1b) is obvious.

In this case we term $A(\pi)$ an order homomorphism of $A(K)$ into $A(K')$. We shall, without further comment, consider π as taking K' into K .

2 Real-valued functions on K.

A real-valued function defined on K is termed convex if $x, y \in K$ and $0 \leq \lambda \leq 1$ imply that $\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y)$. If $-f$ is convex, f is termed concave. Thus if f is both convex and concave then it is affine.

Recall that $f : K \rightarrow \mathbb{R}$ is lower semi-continuous (l.s.c.) if $\{k \in K : f(k) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$, and is upper semi-continuous (u.s.c.) if $-f$ is l.s.c. Thus if f is l.s.c. and u.s.c. then it is continuous.

If $f : K \rightarrow \mathbb{R}$, the supergraph of f , $\text{Sup}(f)$, is the set $\{(k, \alpha) : k \in K, \alpha \geq f(k)\} \subset E \times \mathbb{R}$. The subgraph of f , $\text{Sub}(f)$, is $\{(k, \alpha) : k \in K, \alpha \leq f(k)\} \subset E \times \mathbb{R}$. It is easily verified that f is convex if and only if $\text{Sup}(f)$ is convex, and is concave if and only if $\text{Sub}(f)$ is convex. Also f is l.s.c. if and only if $\text{Sup}(f)$ is closed, and is u.s.c. if and only if $\text{Sub}(f)$ is closed.

We shall first be interested in density results and various maximum principles.

PROPOSITION I.2.1. If K is a compact convex set, the vector space of all differences of continuous convex functions on K is dense in $C(K)$.

Apply the fact that the continuous convex functions form a cone, are closed under finite suprema, and separate the points of K since the continuous affine functions do. Now apply lemma I.1.16 and the Stone-Weierstrass theorem.

PROPOSITION I.2.2. If f is a bounded u.s.c. convex function on K and g is a bounded u.s.c. concave function on K with $f|_{\partial_e K} \leq g|_{\partial_e K}$ then $f \leq g$.

Assume there is $k_0 \in K$ with $g(k_0) < f(k_0)$. The subgraph of g is a closed convex set in $E \times \mathbb{R}$, not containing $(k_0, f(k_0))$. By Hahn-Banach separation there is a closed hyperplane in $E \times \mathbb{R}$ separating $\text{Sub}(g)$ from $(k_0, f(k_0))$. This is the graph of a continuous affine function on E ; let h denote its restriction to K . Then $h > g$, $f(k_0) > h(k_0)$.

Now $f - h$ is convex and u.s.c. The set $\{k \in K : (f - h)(k) = \sup \{(f - h)(k') : k' \in K\}\}$ is a non-empty closed convex subset of K which is easily seen to be a face. This set will have an extreme point, which must also be extreme in K . Hence

$$\begin{aligned} \sup \{(f - h)(k) : k \in K\} &= \sup \{(f - h)(k) : k \in \partial_e K\} \\ &\leq \sup \{(f - g)(k) : k \in \partial_e K\} \\ &\leq 0. \end{aligned}$$

Hence $0 < f(k_0) - h(k_0) \leq 0$ - a contradiction.

COROLLARY I.2.3. If f, g are the difference of bounded u.s.c. affine functions on K and $f|_{\partial_e K} \leq g|_{\partial_e K}$ then $f \leq g$.

Suppose $f = f_1 - f_2$ and $g = g_1 - g_2$ where f_1, f_2, g_1 and g_2 are bounded u.s.c. and affine. By I.2.2 $f_1 + g_2 \leq f_2 + g_1$, so $f = f_1 - f_2 \leq g_1 - g_2 = g$.

PROPOSITION I.2.4. If K is a compact convex set and $f : K \rightarrow \mathbb{R}$ is bounded, l.s.c. and affine, then there is an increasing net from $A(K)$ converging pointwise to f .

Consider the family $\{g \in A(K) : g < f\}$. We first show that the family is upwards filtering. Suppose $g_1, g_2 \in A(K)$ and $g_1, g_2 < f$. Without loss of generality we may assume that all the functions f, g_1 and g_2 are positive. Let $M_1 = \{(k, \alpha) : k \in K, 0 \leq \alpha \leq g_1(k)\}$, and $N = \text{Sup}(f)$. That $g_1, g_2 < f$ implies that $N \cap (M_1 \cup M_2) = \emptyset$. As f is affine,

$N \cap \text{co}(M_1 \cup M_2) = \emptyset$. As M_1 and M_2 are compact as well as convex so is $\text{co}(M_1 \cup M_2)$. Strictly separating $\text{co}(M_1 \cup M_2)$ from N by a closed hyperplane gives us the graph of $h \in A(K)$ with $g_1, g_2 < h < f$. Again if $k_0 \in K$ and $\alpha < f(k_0)$, we may find $h \in A(K)$ so that $h < f$ and $h(k_0) > \alpha$. I.e. $\sup \{g \in A(K) : g < f\} = f$ as required.

We now introduce the notions of upper and lower envelope. Suppose $X \subset K$ with $\partial_e K \subset X$, and that $f : X \rightarrow [\alpha, \infty]$ for some $\alpha \in \mathbb{R}$. The lower envelope of f is the function

$$\check{f} = \sup \{a \in A(K) : a|_X \leq f\}.$$

It is easily verified that $\check{f}$ is that function with supergraph the closed convex hull of the supergraph of f . An argument similar to that used above shows that $\check{f} = \sup \{a \in A(K) : a|_X < f\}$. In particular we note that $\check{f}$ is l.s.c. and convex.

Similarly if $f : X \rightarrow [-\infty, \alpha]$, the upper envelope of f is $\hat{f} = -((-f)^\vee)$, an u.s.c. concave function. A number of trivial results are summarised in:

PROPOSITION I.2.5. Let K be a compact convex set, $\partial_e K \subset X \subset K$ and $f, g : X \rightarrow [-\infty, \alpha]$ for $\alpha \in \mathbb{R}$.

- (1) $\hat{f}(k) \leq \alpha \quad (k \in K)$.
- (2) $f \leq g \Rightarrow \hat{f} \leq \hat{g}$.
- (3) $(f + g)^\wedge \leq \hat{f} + \hat{g}$.
- (4) $(\beta f)^\wedge = \beta(\hat{f})$. (Proof)

In simpler cases the use of the subgraph enables us to compute upper envelopes precisely:

PROPOSITION I.2.6. Let F_j be closed faces of K and let b_j be real-valued functions on K such that $b_j|_{F_j} \in A(F_j)^+$ and

$b_j|_{K \setminus F_j} = 0$ for $j = 1, 2, \dots, m$. Suppose also $a_i \in A(K)^+$ for $i = 1, 2, \dots, n$. Let $f = a_1 \vee a_2 \vee \dots \vee a_n \vee b_1 \vee \dots \vee b_m$. If $k \in K$ there is a convex combination $k = \sum_{i=1}^n \lambda_i x_i + \sum_{j=1}^m \mu_j y_j$, with $y_j \in F_j$, such that

$$\hat{f}(k) = \sum_{i=1}^n \lambda_i a_i(x_i) + \sum_{j=1}^m \mu_j b_j(y_j).$$

Define $Q_i = \{(k, \alpha) : k \in K, 0 \leq \alpha \leq a_i(k)\}$

$$R_j = \{(k, \alpha) : k \in K, 0 \leq \alpha \leq b_j(k)\}.$$

Then the subgraph of f is the set $\bigcup_{i=1}^n Q_i \cup \bigcup_{j=1}^m R_j$. As Q_i and R_j are compact $\text{co}(\text{sub}(f))$ is also compact, hence closed, so is equal to $\text{sub}(f)$. The result is now immediate.

Finally we note:

PROPOSITION I.2.7. Let $(f_i)_{i \in I}$ be a downward filtering family of u.s.c. functions on K such that $\inf_{i \in I} f_i(k)$ is finite for all $k \in K$. Then

$$(\inf_{i \in I} f_i)^\wedge = \inf_{i \in I} (f_i)^\wedge.$$

It is clear that $(\inf f_i)^\wedge \leq \inf (f_i)^\wedge$. Suppose $a \in A(K)$ and $a > \inf f_i$. It will suffice to show that there is $j \in I$ with $a > f_j$. By assumption for each $k \in K$ there is $i_k \in I$ with $a(k) > f_{i_k}(k)$. By u.s.c. of f_{i_k} this strict inequality persists in an open neighbourhood U_k of k . Compactness now gives us a finite cover $U_{k_1}, \dots, U_{k_n}$ of K . Choosing f_j smaller than each $f_{i_{k_s}}$ provides the desired function.

3 Representation of points by measures.

We have seen in I.1.9 that if $\mu \in P(K)$ there is a unique $k \in K$ such that $\int a d\mu = a(k)$ whenever $a \in A(K)$. This result can be extended considerably. We shall content ourselves with the following result.

We denote by $A_s(K)$ the uniform closure, in the space of all bounded functions on K , of the vector space of all differences of bounded l.s.c. affine functions on K .

PROPOSITION I.3.1. If $f \in A_s(K)$ and $\mu \in P(K)$ then $\int f d\mu = f(r(\mu))$.

Suppose first that f is bounded, l.s.c. and affine, then by I.2.4 there is an increasing net $f_\gamma \nearrow f$ from $A(K)$. We then have:

$$f(r(\mu)) = \bigvee_\gamma f_\gamma(r(\mu)) = \bigvee_\gamma \mu(f_\gamma) = \mu(\bigvee_\gamma f_\gamma) = \mu(f),$$

where we write $\mu(h)$ for $\int h d\mu$. Clearly this will persist for differences of such functions. That it persists for the closure of this space is an immediate consequence of the Dominated Convergence Theorem.

We state for reference a few facts about $A_s(K)$.

PROPOSITION I.3.2. If $f \in A_s(K)$ then:

- (1) $f \geq 0 \iff f|_{\partial_e K} \geq 0$.
- (2) $\|f\| = \sup\{|f(k)| : k \in \partial_e K\}$.

These are immediate consequences of Corollary I.2.3 for differences of bounded l.s.c. affine functions. The general case follows from this by a limiting argument.

We are interested in finding probability measures with a given resultant that are supported (as far as possible) by the set of extreme points of K . As an aid to this we define an order on the space of all regular Borel measures on K , $M(K)$, by

$\mu \prec \nu$ if $\mu(f) \leq \nu(f)$ whenever f is continuous and convex.

In particular we note that if $\nu \in P(K)$ and either $\mu \succ \nu$ or $\mu \prec \nu$ then $\mu \in P(K)$ and $r(\mu) = r(\nu)$. For if $a \in A(K)$ then a and $-a$ are continuous and convex so $\mu(a) \leq \nu(a) = -\nu(-a) \leq -\mu(-a) = \mu(a)$.

PROPOSITION I.3.3. Let $\mu, \nu \in M(K)$ then $\mu \prec \nu$ if and only if $\nu(f) \leq \mu(\hat{f})$ whenever $f \in C(K)$.

An argument similar to that used in I.2.4 shows that $\hat{f}$ is the pointwise infimum of a directed family of finite infima of elements of $A(K)$; i.e. $\hat{f}$ is the infimum of a family of concave continuous functions. It follows that if $\mu \prec \nu$ then $\mu(\hat{f}) \geq \nu(\hat{f}) \geq \nu(f)$.

On the other hand choosing f continuous and concave, $\hat{f} = f$ and $\nu(f) \leq \mu(f)$. This implies $\mu \prec \nu$ as claimed.

Our first result on the existence of measures is:

PROPOSITION I.3.4. Let K be a compact convex set, $f \in C(K)$ and $\mu \in P(K)$. Then there is $\nu \in P(K)$ with $\nu \succ \mu$ and $\nu(f) = \mu(\hat{f})$.

By I.2.5 $\bar{\Phi} : g \rightarrow \mu(\hat{g})$ is a sublinear functional on $C(K)$. Define a linear functional on $\mathbb{R}f$, ν_0 , by $\nu_0(\alpha f) = \alpha \mu(\hat{f})$. If $\alpha \geq 0$ then $\nu_0(\alpha f) = \alpha \mu(\hat{f}) = \mu(\alpha \hat{f}) = \bar{\Phi}(\alpha f)$. On the other hand if $\alpha < 0$ setting $\beta = -\alpha$ and noting that $-\mu(\widehat{\beta f}) \leq \mu(\widehat{-\beta f})$ we have $\nu_0(\alpha f) = -\beta \mu(\hat{f}) = -\mu(\widehat{\beta f}) \leq \mu(\widehat{-\beta f}) = \mu(\widehat{\alpha f}) = \bar{\Phi}(\alpha f)$. Hence ν_0 is majorised by $\bar{\Phi}$. Extend ν_0 by the Hahn-Banach theorem to a linear functional ν on $C(K)$ majorised by $\bar{\Phi}$. Then $\nu(f) = \mu(\hat{f})$ and $\nu(g) \leq \mu(\hat{g})$ for all $g \in C(K)$.

If $g \in C(K)$ and $g \leq 0$, then $\hat{g} \leq 0$ so $\nu(g) \leq \mu(\hat{g}) \leq 0$. So ν is positive. Finally as $\nu(g) \leq \mu(\hat{g})$ for all $g \in C(K)$, I.3.2 implies that $\nu \succ \mu$. By the comment made above it follows that

$\nu \in P(K)$. The proof is now complete.

As an application of this we have:

PROPOSITION I.3.5. If $k \in K$ the following are equivalent:

- (1) $k \in \partial_e K$.
- (2) $f(k) = \hat{f}(k)$ for all $f \in C(K)$.
- (3) $f(k) = \hat{f}(k)$ for all $f : K \rightarrow [-\infty, \infty)$ which are

u.s.c. and bounded.

(1) $\implies$ (2). If $k \in \partial_e K$ and $f \in C(K)$, I.3.4 implies that there is $\nu \in P(K)$ with $\nu(f) = \hat{f}(k)$ and $\nu \succ \xi_k$. In particular ν has resultant k , so $\nu = \xi_k$ by I.1.10. I.e. $f(k) = \hat{f}(k)$.

(2) $\implies$ (3). Let f_α be a net from $C(K)$ filtering downward with infimum f . Then $\inf f_\alpha$ is u.s.c. and concave, so dominates $\hat{f}$. Thus $f(k) = \inf f_\alpha(k) = \inf \hat{f}_\alpha(k) \geq \hat{f}(k) \geq f(k)$.

(3) $\implies$ (2). a fortiori.

(2) $\implies$ (1). Considering $-f$ we have $f(k) = \check{f}(k)$. Let $\mu \in P(K)$ and $r(\mu) = k$. By I.3.3 we have

$$f(k) = \check{f}(k) \leq \mu(f) \leq \hat{f}(k) = f(k),$$

since $\mu \succ \xi_k$. On the other hand this also implies $\xi_k \succ \mu$ again by I.3.3. Since the order on $P(K)$ is proper (because of I.2.1), $\xi_k = \mu$. Now k is extreme by I.1.10.

COROLLARY I.3.6. If $\partial_e K \subset X \subset K$ and $f : X \rightarrow \mathbb{R}$ is bounded and continuous, then $f(k) = \hat{f}(k) = \check{f}(k)$ for all $k \in \partial_e K$.

Let $f \geq \alpha$, extend f to an u.s.c. function f' on K by:

$$\begin{aligned} f'(k) &= \lim_{x \in X} \sup_{x \rightarrow k} f(x) & (k \in \bar{X}) \\ &= \alpha & (k \in K \setminus \bar{X}). \end{aligned}$$

Clearly if $a \in A(K)$ and $a|_X \geq f$ then $a \geq f'$. Thus $\hat{f} = \hat{f}'$, and the result follows from I.3.5 immediately.

If $f \in C(K)$, we define the boundary set of f , B_f , to be $\{k \in K : f(k) = \hat{f}(k)\}$.

COROLLARY I.3.7. $\partial_e K = \bigcap \{B_f : f \in C(K)\}$.

We shall be interested in measures in $P(K)$ that are maximal with respect to $<$.

PROPOSITION I.3.8. If $\mu \in P(K)$ the following are equivalent:

- (1) μ is maximal for $<$.
- (2) $\mu(\hat{f}) = \mu(f)$ for all $f \in C(K)$.
- (3) $\mu(\hat{f}) = \mu(f)$ for all convex $f \in C(K)$.

(1) $\implies$ (2). By I.3.4 there is $\nu \in P(K)$ with $\nu > \mu$ and $\nu(f) = \mu(\hat{f})$. By maximality, $\mu = \nu$ so $\mu(f) = \mu(\hat{f})$.

(2) $\implies$ (3). a fortiori.

(3) $\implies$ (1). Suppose f is continuous and convex, and let $\nu > \mu$. Then $f = \check{f}$, so by I.3.3

$$\mu(f) = \mu(\check{f}) \leq \nu(f) \leq \mu(\hat{f}) = \mu(f).$$

Thus μ and ν coincide on continuous convex functions, and thence on $C(K)$ in virtue of I.2.1.

COROLLARY I.3.9. If μ is $>$ -maximal then $\mu(\overline{\partial_e K}) = 1$.

Let C be a compact subset of K such that $C \cap \overline{\partial_e K} = \emptyset$. Let $f \in C(K)$ with $f|_{\overline{\partial_e K}} = 0$, $f|_C = -1$ and $-1 \leq f \leq 0$. As $\hat{f} = 0$ we have

$$0 = \mu(\hat{f}) = \mu(f) \leq -\mu(C) \leq 0.$$

It now follows from regularity that $\mu(\overline{\partial_e K}) = 1$.

It follows from (2) of I.3.8 that μ is maximal if and only if $\mu(B_f) = 1$ for each $f \in C(K)$. For this reason maximal measures are also termed boundary measures.

THEOREM I.3.10. If $k \in K$ there is $\mu \in P(K)$ that is maximal for $>$ and with $r(\mu) = k$.

The proof applies Zorn's lemma on the set $\{\mu \in P(K) : \mu \succ \varepsilon_k\}$.

We must show that any chain $\{\mu_\alpha\}$ in this set has an upper bound. Let μ_0 be any weak* accumulation point of $\{\mu_\alpha\}$ - by compactness of $P(K)$. Choose α and a continuous convex f . For every $\varepsilon > 0$ there exists $\beta > \alpha$ with

$$|\mu_\beta(f) - \mu_0(f)| < \varepsilon.$$

As $\mu_\alpha < \mu_\beta$, $\mu_\alpha(f) \leq \mu_\beta(f)$ so $\mu_\alpha(f) \leq \mu_0(f) + \varepsilon$. As ε was arbitrary, $\mu_\alpha(f) \leq \mu_0(f)$. Hence $\mu_\alpha \leq \mu_0$, so μ_0 is an upper bound for $\{\mu_\alpha\}$ as required.

If $k \in K$, we denote by $\underline{P}_k(K) = \{\mu \in P(K) : r(\mu) = k\}$.

The set $\underline{Q}(K)$ of all maximal measures in $P(K)$ is easily seen to be a face of $P(K)$, using (2) of I.3.8. We define $\underline{Q}_k(K) = \underline{P}_k(K) \cap \underline{Q}(K)$.

Theorem I.3.10 asserts that $\underline{Q}_k(K) \neq \emptyset$ if $k \in K$.

In the metrisable case we have a useful characterisation of maximal measures which is not valid in general.

THEOREM I.3.11. If K is a metrisable compact convex set, $\partial_e K$ is a G_δ and $\mu \in P(K)$ is maximal if and only if $\mu(\partial_e K) = 1$.

It will suffice to find $f \in C(K)$ with $B_f = \partial_e K$. By I.1.17 and I.2.1 there is a sequence $\{g_n^1 - g_n^2\}$ dense in $C(K)$, where g_n^1 is continuous and convex. Let us define $f \in C(K)$ by

$$f = \sum_{i=1}^2 \sum_{n=1}^{\infty} 2^{-n} \|g_n^i\|^{-1} g_n^i.$$

To see that $B_f = \partial_e K$, suppose $k \in K \setminus \partial_e K$, so that $k = \lambda x + (1 - \lambda)y$ with $x, y \in K$, $x \neq y$ and $0 < \lambda < 1$. By density of the $\{g_n^1 - g_n^2\}$ there are i, n with $g_n^i(k) < \lambda g_n^i(x) + (1 - \lambda)g_n^i(y)$ or else $h(k) = \lambda h(x) + (1 - \lambda)h(y)$ for all $h \in C(K)$, which is clearly not the case if we take $h = a^2$ for $a \in A(K)$ with $a(x) \neq a(y)$.

Now certainly

$$\begin{aligned} f(k) &< \lambda f(x) + (1 - \lambda)f(y) \\ &\leq \lambda \hat{f}(x) + (1 - \lambda)\hat{f}(y) \\ &\leq \hat{f}(k), \end{aligned}$$

so $k \notin B_f$.

COROLLARY I.3.12. If K is a metrisable compact convex set and $k \in K$ there is $\mu \in P(K)$ with $r(\mu) = k$ and $\mu(\partial_e K) = 1$.

One notion of interest to us is that of a dilated set. A Borel subset of K , D , is dilated if $\mu \in P(K)$ and $r(\mu) \in D$ imply that $\mu(D) = 1$.

PROPOSITION I.3.13. If F is a closed convex subset of a compact convex set K , then F is dilated if and only if F is a face of K .

If F is dilated and $x, y \in K$, $0 < \lambda < 1$ with $\lambda x + (1 - \lambda)y \in F$ then $r(\lambda \varepsilon_x + (1 - \lambda) \varepsilon_y) \in F$. It follows that $x, y \in F$ as F is dilated.

Now suppose F is a face. The set $\{\mu \in P(K) : \mu(F) = 1\}$ is easily seen to coincide with $\bigcap \{\mu \in P(K) : \mu(a) = 1\}$ where the intersection is over all $a \in C(K)$ such that $a|_F = 1$, $a \leq 1$. Thus $\{\mu \in P(K) : \mu(F) = 1\}$ is closed. As the resultant map is continuous and affine, $r^{-1}(F)$ is a closed face of $P(K)$. By the Krein-Milman theorem, $r^{-1}(F) = \overline{\text{co}} \{\varepsilon_k : k \in F\}$. As $\{\varepsilon_k : k \in F\} \subset \{\mu \in P(K) : \mu(F) = 1\}$ and this latter set is closed and convex, it follows that $r^{-1}(F) \subset \{\mu \in P(K) : \mu(F) = 1\}$.

A subset G of a convex set C is a σ -face if $x, y \in C$, $0 < \lambda < 1$ and $\lambda x + (1 - \lambda)y \in G$ imply that $x, y \in G$.

LEMMA I.3.14. A subset G of a compact convex set K is a σ -face if and only if it is a union of faces of K . If G is a closed σ -face then it is a union of closed faces of K .

It is easily verified that a union of faces of K is a σ -face. Suppose conversely that G is a σ -face of K . If $k \in G$, let F be a maximal convex subset of G containing k . We claim that F is a face of K .

Suppose $f \in F$, $0 < \lambda < 1$ and $f = \lambda x + (1 - \lambda)y$ with $x \notin F$.

We claim $\text{co}(\{x\} \cup F) \subset G$ which will contradict the maximality of F . Suppose $0 < \alpha < 1$ and $k = \alpha x + (1 - \alpha)f \notin G$. Writing

$$\gamma = \frac{\lambda}{(1 - \alpha)\lambda + \alpha}$$

we have $0 < \gamma < 1$ and $f = \gamma k + (1 - \gamma)y$. As $f \in G$ and G is a σ -face, $k \in G$ contradicting our assumption.

If G is closed, and F is maximal as a convex subset of G , then $\overline{F} \subset G$ is also convex, so $\overline{F} = F$ is closed.

COROLLARY I.3.15. If G is a closed subset of a compact convex set K , then G is dilated if and only if G is a σ -face.

If G is a σ -face, $r(\mu) \in G$ and F is a closed face of K with $r(\mu) \in F \subset G$, then $\text{supp}(\mu) \subset F \subset G$. Conversely if G is dilated the proof of I.3.13 suffices to show that G is a σ -face.

PROPOSITION I.3.16. Let $f : K \rightarrow \mathbb{R}$ be an u.s.c. positive affine function, then $f^{-1}(0)$ is dilated.

Using I.3.1 if $r(\mu) \in f^{-1}(0)$ then $\mu(f) = f(r(\mu)) = 0$.

As $f \geq 0$ it follows that $\mu(f^{-1}(0)) = 1$.

CHAPTER II

SIMPLEXES

1 Basic properties.

The spaces of continuous affine functions on a compact simplex share many of the properties of spaces of continuous functions on a compact Hausdorff space. In this section we prove the basic results on compact simplexes.

We first need a result on the upper envelope of a class of functions:

PROPOSITION II.1.1. Let S be a compact simplex and f a bounded u.s.c. convex function on S . Then

$$\hat{f}(k) = \inf \{a(k) : a \in A(K), a \geq f\}$$

is affine on S .

Note first that the space of all bounded affine functions on S may be identified with $A(S)^{**}$, and is therefore a lattice (cf. I.1.13). Also that the pointwise infimum of a downward directed family from $A(S)^{**}$ will be affine. It will suffice to show that f is the pointwise infimum of such a family. We take for this family the collection, U , of all finite infima (in $A(S)^{**}$) of elements of $A(S)$ dominating f .

If $a \in A(S)^{**}$ and is u.s.c. on S , then the formula

$$a^+(k) = \sup \{a(x) : 0 \leq x \leq k\}$$

for $k \in S$ shows that a^+ is also u.s.c. The formula $a_1 \wedge a_2 = a_2 - (a_2 - a_1)^+$ now shows that each $a \in U$ is l.s.c. We now see that U is a downward directed family of l.s.c. functions whose infimum, z , satisfies $z \leq \hat{f}$.

If $a \in U$ and $\varepsilon > 0$ then $a + \varepsilon$ is l.s.c. and affine and is therefore the pointwise supremum of the family, V , of functions $v \in A(S)$ such that $v < a + \varepsilon$. The family V is upwards

directed with supremum $a + \xi \geq f + \xi$. Since f is u.s.c. a compactness argument shows that there is $w \in V$ such that $f < w < a + \xi$. Therefore $\hat{f} < a + \xi$ for each $a \in U$ and all $\xi > 0$, and hence $\hat{f} = z$.

As a first application we derive the important uniqueness theorem.

THEOREM II.1.2. If K is a compact convex set the following are equivalent:

- (1) K is a simplex.
- (2) There is a linear operator $E : C(K) \rightarrow A_s(K)$ such that $E \geq 0$ and $Ea = a$ if $a \in A(K)$.
- (3) Every point of K is the resultant of a unique boundary measure.
- (4) There is an affine map $m : K \rightarrow P(K)$ with $r(m(k)) = k$.

(1) $\implies$ (2). Note first that $f \mapsto \hat{f}$ is additive on the cone of continuous convex functions on K . For if $k \in K$, let μ be a boundary measure representing k , so that $\mu(f) = \mu(\hat{f})$ by I.3.8. As f is affine (II.1.1) $\mu(\hat{f}) = \hat{f}(k)$ by I.3.1. Thus if g is also continuous and affine then $(f + g)^\wedge(k) = \mu(f + g) = \mu(f) + \mu(g) = \hat{f}(k) + \hat{g}(k)$. Hence $(f + g)^\wedge = \hat{f} + \hat{g}$.

If f is the difference of two continuous convex functions f_1 and f_2 , we define $Ef = f_1 - f_2$. By the additivity of $f \mapsto \hat{f}$, E is well-defined and linear. From I.2.5 it follows that $\|E(f - g)\| \leq \|f - g\|$ for all continuous convex f and g . By the density of such differences in $C(K)$ and the completeness of $A_s(K)$ E may be extended to the whole of $C(K)$ by continuity. Clearly if $a \in A(K)$ then $Ea = a$.

If $f_1 - f_2 \geq 0$ (f_1 continuous and convex), then $f_1 \geq f_2$ so $\hat{f}_1 \geq \hat{f}_2$ (I.2.5), i.e. $\hat{f}_1 - \hat{f}_2 \geq 0$ so E is positive on this subspace. The extension to the whole of $C(K)$ is positive because

the positive cone in $A_s(K)$ is closed.

(2) $\implies$ (3). If $k \in K$, define $\lambda(f) = (Ef)(k)$. Then λ is a positive linear functional on $C(K)$ taking 1_K to 1. We may identify λ with a probability measure on K that we also denote by λ . Suppose $\mu \in P(K)$ and $r(\mu) = k$. If f is continuous and convex, then

$$\lambda(f) = (Ef)(k) = \mu(Ef).$$

But $f = \sup \{a \in A(K) : a < f\}$ (cf. proof of I.2.4), so $Ef \geq \sup \{Ea : a < f\} = \sup \{a : a < f\} = f$. Thus $\mu(Ef) \geq \mu(f)$. I.e. we know $\lambda \geq \mu$. It follows that λ is the unique maximal measure representing k .

(3) $\implies$ (4). The map taking k to its unique representing boundary measure has all the desired properties.

(4) $\implies$ (1). By I.1.11 r extends to a positive linear operator $R : C(K)^* \longrightarrow A(K)^*$, and m to a positive linear $M : A(K)^* \longrightarrow C(K)^*$ with $R(Ma) = a$ for all $a \in A(K)^*$. If $a, b \in A(K)^*$, $R(Ma \vee Mb) \geq RMa, RMb$ so exceeds a and b . On the other hand if $c \geq a, b$ then $Mc \geq Ma, Mb$ so $c = R(Mc) \geq R(Ma \vee Mb)$. It follows that $A(K)^*$ is a lattice, so K is a simplex.

A measure μ on K is an affine dependence if $\mu(a) = 0$ for all $a \in A(K)$. The uniqueness result may be restated as :

COROLLARY II.1.3. If S is a simplex then there are no non-zero boundary affine dependences on S .

We shall denote by μ_k the unique boundary measure representing k whenever S is a simplex. A consequence of II.1.1 that we shall need is:

COROLLARY II.1.4. Let S be a compact simplex, f a bounded u.s.c. convex function, and g a bounded l.s.c. concave function. If $f \leq g$ then $\hat{f} \leq \check{g}$, and if $f < g$ then $\hat{f} < \check{g}$.

$\check{g} - \hat{f}$ is l.s.c. and affine, so attains its infimum on $\partial_e S$ (by the Krein-Milman theorem). But on $\partial_e S$, $\hat{f}(k) = f(k)$ and $\check{g}(k) = g(k)$, so the desired result follows.

THEOREM II.1.5. (Edwards) Let S be a compact simplex, $f : S \rightarrow [-\infty, \infty)$ be u.s.c. and convex, and $g : S \rightarrow (-\infty, \infty]$ be l.s.c. and concave. If $f \leq g$ there is $h \in A(S)$ with $f \leq h \leq g$.

Note first that we may suppose f and g are bounded, for if not we can replace f by $f \vee (\inf_{k \in S} g(k))$ and g by $g \wedge (\sup_{k \in S} f(k))$ without altering the assumptions of the theorem. A separator for these two will automatically separate f and g .

By an inductive argument we shall construct a sequence $\{h_n\}_{n=0}$ from $A(S)$ such that for $n \geq 1$

$$f - 1 < h_0 < g + 1$$

$$(h_{n-1} - 2^{-n}) \vee (f - 2^{-n}) < h_n < (h_{n-1} + 2^{-n}) \vee (g + 2^{-n}),$$

and observe that this will establish the result as $\{h_n\}$ will be a Cauchy sequence, the limit of which satisfies the requirements of the theorem.

For $n = 0$, we have $(f - 1)^\wedge < (g + 1)^\vee$ by Corollary II.1.4, and the existence of h_0 follows by separation in product spaces using the Hahn-Banach theorem. The existence of h_n follows similarly when the induction hypothesis is used to verify that:

$$(h_{n-1} - 2^{-n}) \vee (f - 2^{-n}) < (h_{n-1} + 2^{-n}) \wedge (g + 2^{-n}).$$

The second result that we shall prove is a generalisation of this. Before proving it we need more notation.

If C is a convex set and E a vector space, a map

$\Phi : C \longrightarrow 2^E$ is termed affine if, for every $c \in C$, $\Phi(c)$ is a non-empty convex subset of E , and such that:

$$\lambda \Phi(c_1) + (1 - \lambda) \Phi(c_2) \subset \Phi(\lambda c_1 + (1 - \lambda)c_2)$$

whenever $0 < \lambda < 1$ and $c_1, c_2 \in C$.

If X, Y are topological spaces, a map $\Phi : X \longrightarrow 2^Y$ is termed lower semi-continuous if whenever $U \subset Y$ is open, then $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is open in X .

In our context we shall deal with maps from a compact simplex S into 2^E , where E is a topological vector space, which are both l.s.c. and affine. If Φ is such a map then so are the following:

$$\overline{\Phi} : k \longmapsto \overline{\Phi(k)}$$

$$(\Phi + U) : k \longmapsto \Phi(k) + U \text{ where } U \text{ is a convex open neighbourhood of } 0 \text{ in } E,$$

$$T \circ \Phi : k \longmapsto T(\Phi(k)) \text{ where } T : E \longrightarrow F \text{ is a continuous linear operator from } E \text{ into a topological vector space } F.$$

Not quite so obvious, but more useful, is the following result:

LEMMA II.1.6. Let C be a convex subset of a topological vector space and S, T l.s.c. affine maps of $C \longrightarrow 2^E, 2^F$ respectively, where E and F are topological vector spaces. Let $f : E \longrightarrow F$ be continuous and linear, and let U be a non-empty open convex set in F . If $x \in C$ define

$$Rx = Sx \cap f^{-1}(Tx + U).$$

If $Rx \neq \emptyset$ for all $x \in C$, then R is l.s.c. and affine.

That R is affine is easily checked. Let $\emptyset \neq V$ be open in E , and let $W = \{(a, b) \in E \times F : f(a) - b \in U\}$, so that W is open in $E \times F$. Let $S \times T$ denote the map $x \longmapsto Sx \times Tx$, then it is clear that $S \times T$ is l.s.c. Now the fact that R is l.s.c. is clear

from the equality:

$$\{x \in C : Rx \cap V \neq \emptyset\} = \{x \in C : (S \times T)(x) \cap W \cap (V \times F) \neq \emptyset\}.$$

Before proving the main selection theorems, we show the existence of "approximate selections".

PROPOSITION II.1.7. Let S be a compact simplex and E a locally convex space. Suppose $\Phi : S \rightarrow 2^E$ is l.s.c. and affine. If U is a convex neighbourhood of the origin in E , then let $(\Phi + U)(k) = \Phi(k) + U$ for all $k \in S$. There exists a continuous affine selection for $\Phi + U$.

We prove this first for E finite dimensional. Applying Lemma II.1.6 we see that $\Phi + U$ is l.s.c. and affine, and hence that $\overline{\Phi + U}$ is also. If $E = \mathbb{R}$, define $f(k) = \inf \{r \in \mathbb{R} : r \in \overline{\Phi(k) + U}\}$ and $g(k) = \sup \{r \in \mathbb{R} : r \in \overline{\Phi(k) + U}\}$. That f is convex and g is concave is immediate from the affineness of $\overline{\Phi + U}$. We also have that f is u.s.c. for $f^{-1}(-\infty, \alpha) = \{k \in S : (\overline{\Phi(k) + U}) \cap (-\infty, \alpha) \neq \emptyset\}$ is open, and similarly g is l.s.c. By Edwards' theorem there is a continuous affine separator, h , for these which will be a selection for $\overline{\Phi + U}$. Noting that $\overline{\Phi + U}/2 \subset \Phi + U$ completes the argument for $E = \mathbb{R}$.

Suppose $\dim(E) > 1$, and let $E = \mathbb{R} \times F$ with p, q the canonical projections onto $\mathbb{R}, F$ respectively. Suppose A, W are convex symmetric neighbourhoods of the origin in $\mathbb{R}, F$ respectively with $\overline{A \times W} \subset U$. The comments above show that $p\Phi : k \mapsto p(\overline{\Phi(k)})$ is l.s.c. and affine, and the argument above shows that there is a continuous affine selection for $p\Phi$, f . If we now set $\Psi = q \circ (\Phi \cap p^{-1}(f + A))$, lemma II.1.6 shows that Ψ is l.s.c. and affine. By induction we may suppose there is a continuous affine selection for $\Psi + W$, g . It is clear

that (f, g) is a continuous affine selection for $\bar{\Phi} + U$.

In general, suppose U is convex, symmetric and open. If $y \in E$ let

$$Uy = \{k \in S : \bar{\Phi}(k) \cap (y + U/2) \neq \emptyset\}$$

which is open. The sets cover S which is compact, so there is a finite-dimensional subspace H of E such that the sets Uy cover S as y ranges over H . If $W = H \cap U/2$ and $\bar{\Psi} = (\bar{\Phi} + U/2) \cap H$, then $\bar{\Psi}$ is affine and l.s.c. Now any continuous affine selection for $\bar{\Psi} + W$ will also be a selection of $\bar{\Phi} + U$.

THEOREM II.1.8. (Lazar) Let S be a compact simplex, and E a Frechet space. Let $\bar{\Phi} : S \rightarrow 2^E$ be l.s.c., affine and suppose $\bar{\Phi}(k)$ is closed for all $k \in S$. Then there is a continuous affine selection for $\bar{\Phi}$.

Let p_n be an increasing sequence of semi-norms defining the topology of E . Define a sequence of continuous affine selections as follows:

$$h_1(k) \in \bar{\Phi}(k) + \{x \in E : p_1(x, 0) < 2^{-1}\}$$

and if h_{n-1} is defined,

$$h_n(k) \in \bar{\Phi}(k) \cap \{x \in E : p_{n-1}(x, h_{n-1}(k)) < 2^{-n+1}\} \\ + \{x \in E : p_n(x, 0) < 2^{-n}\},$$

this being possible as the intersection is non-empty, for the semi-norms are increasing. The sequence h_n is Cauchy so converges to a continuous affine function from S to E , which is a selection as $\bar{\Phi}(k)$ is closed for each $k \in S$.

At the expense of a more complicated statement we can improve this result slightly.

THEOREM II.1.9. Let S be a compact simplex and E a Frechet space. Let $\Phi : S \rightarrow 2^E$ be affine and let $\Phi(k)$ be closed for every $k \in S$. Suppose also that for every convex symmetric open neighbourhood of 0 in E , U , there is $\Phi_U : S \rightarrow 2^E$ which is affine and l.s.c. and satisfies $\Phi(k) \subset \Phi_U(k) \subset \Phi(k) + U$ for each $k \in S$. Then there is a continuous affine selection for Φ .

Let p_n be an increasing sequence of semi-norms defining the topology of E , and let $U_{n,\beta} = \{z \in E : p_n(z,0) < \beta\}$, and let $\Phi_{n,\beta}$ be l.s.c. and affine with $\Phi \subset \Phi_{n,\beta} \subset \Phi + U_{n,\beta}$. Fix $\alpha > 1$. If h is a selection for $(\Phi + U_{n,\beta})^-$, then $\Phi_{n+1,\beta/2}(k) \cap (h(k) + U_{n,\alpha\beta}) \neq \emptyset$ for each $k \in S$. By lemma II.1.6 this latter map is l.s.c. and affine, so its closure has a continuous affine selection which will also be a selection for $(\Phi(k) + U_{n+1,\beta/2}) \cap (h(k) + U_{n,\alpha\beta})^-$.

Now define a sequence $\{h_n\}_{n=1}^\infty \subset A(S,E)$ as follows:

$$h_1(k) \in (\Phi(k) + U_{1,1/2})^-$$

and, if h_n is defined,

$$h_{n+1}(k) \in ((h_n(k) + U_{n,\alpha 2^{-n}}) \cap (\Phi(k) + U_{n+1,2^{-n-1}}))^- \subset (\Phi(k) + U_{n+1,2^{-n-1}})^-.$$

It is clear that (h_n) is Cauchy, and that its limit will have values in Φ .

If E is a Banach space, the criterion is rather simpler. Let B_α denote the open ball in E , centre the origin and radius α .

COROLLARY II.1.10. Let S be a compact simplex and E a Banach space. Let $\Phi : S \rightarrow 2^E$ be affine and let $\Phi(k)$ be closed for all $k \in S$. Suppose also that for each $\beta > 0$ there is $\Phi_\beta : S \rightarrow 2^E$ which is affine, l.s.c. and satisfies $\Phi \subset \Phi_\beta \subset \Phi + B_\beta$. Then there is a continuous affine selection for Φ .

2 Bauer simplexes.

We shall look in this section at one special case that has inspired much of the general theory. If $\mathcal{N}$ is a compact Hausdorff space, then $M(\mathcal{N})$ is a lattice, so the base of the positive cone in it, $P(\mathcal{N})$, is a simplex. When endowed with the weak* topology as the dual of $C(\mathcal{N})$, this simplex is compact. The set of extreme points of $P(\mathcal{N})$ is precisely the set of evaluation measures at points of $\mathcal{N}$, so is compact and hence closed. Thus such simplexes are of a special nature. In fact the property of having closed extreme boundary characterises such sets of probability measures amongst compact simplexes.

We define a simplex S to be a Bauer simplex if $\partial_e S$ is closed in S .

THEOREM II.2.1. If K is a compact convex set, the following are equivalent:

- (1) K is a Bauer simplex.
- (2) Every point of K is the barycentre of a unique probability measure supported by $\overline{\partial_e K}$.
- (3) K is a simplex and $x \mapsto \mu_x$ is a continuous map from K into $P(K)$ endowed with the weak* topology as the dual of $C(K)$.
- (4) $\hat{f} \in A(K)$ for all continuous convex functions f defined on K .
- (5) $A(K)$ is a lattice.

(1) $\implies$ (2). This is because now μ is supported by $\partial_e K = \overline{\partial_e K}$ if and only if μ is a boundary measure. The uniqueness theorem for simplexes completes the argument.

(2) $\implies$ (3). That K is a simplex is because the boundary measures for K are certainly supported by $\overline{\partial_e K}$, so are certainly

unique if (2) holds. Also $\mu \mapsto r(\mu)$ is $(1,1)$ and continuous from $P(\overline{\partial_e K})$ onto K . The inverse map, $x \mapsto \mu_x$ is thus continuous.

(3) $\Rightarrow$ (4). $\hat{f}$ is affine by II.1.1. As $\hat{f}(k) = \mu_k(f)$, the map $k \mapsto \hat{f}(k)$ may be decomposed into $k \mapsto \mu_k \mapsto \mu_k(f)$, and is therefore continuous.

(4) $\Rightarrow$ (5). If $a_1, a_2 \in A(K)$, the definition of upper envelopes makes it plain that $\widehat{a_1 \vee a_2}$ is the least upper bound of a_1 and a_2 in $A(K)$, where $a_1 \vee a_2$ is calculated in $C(K)$.

(5) $\Rightarrow$ (1). That K is a simplex is because $A(K)$ is a lattice, hence has the R.D.P. and hence $A(K)^*$ is a lattice. It remains to show that $\partial_e K$ is closed. Suppose $k_1 \notin \partial_e K$, let $k_1 = (1/2)m + (1/2)n$. By the Hahn-Banach theorem we can find $a_1, a_2 \in A(K)$ with

$$a_1(m) = a_2(n) = 1, \quad a_1(n) = a_2(m) = 0.$$

If $a_1 \vee a_2$ is calculated in $C(K)$, then $(a_1 \vee a_2)(k_1) = 1/2$, yet $(a_1 \vee a_2)^\wedge(k_1) \geq 1$ so $k_1 \notin B_f = \{k \in K : \hat{f}(k) = f(k)\}$ where $f = a_1 \vee a_2$. As $\partial_e K \subset B_f$ for all continuous f we see that $\partial_e K = \bigcap \{B_f : f = a_1 \vee a_2 \dots \vee a_n, a_i \in A(K)\}$. If the supremum of a_1, a_2 exists in $A(K)$ it must be equal to $\widehat{a_1 \vee a_2}$ which is therefore continuous. Thus (5) tells us that each B_f , and hence $\partial_e K$, is closed.

COROLLARY II.2.2. The Bauer simplexes, S , are precisely those compact simplexes affinely homeomorphic to a set of probability measures on a compact Hausdorff space endowed with the weak* topology. Namely S may be identified with $P(\partial_e S)$.

An alternative characterisation is by means of extension properties for real valued functions defined on the extreme boundary.

THEOREM II.2.3. If K is a compact convex set, the following are equivalent:

- (1) K is a Bauer simplex.
- (2) Every continuous function $f : \partial_e K \rightarrow \mathbb{R}$ can be extended to a function of $A(K)$.
- (3) Every bounded continuous function $f : \partial_e K \rightarrow \mathbb{R}$ can be extended to a function of $A(K)$.
- (4) Every uniformly continuous function $f : \partial_e K \rightarrow \mathbb{R}$ can be extended to a function of $A(K)$.

Moreover if K is a Bauer simplex the extension of (2) is unique and norm preserving.

- (1) $\implies$ (2). $k \mapsto \mu_k(f)$ is the desired extension.
- (2) $\implies$ (3) is trivial.
- (3) $\implies$ (4) is also trivial because uniformly continuous functions are bounded.
- (4) $\implies$ (1). Let $a_1, a_2 \in A(K)$, and let $a \in A(K)$ extend $a_1 \vee a_2|_{\partial_e K}$. Clearly a is the l.u.b. in $A(K)$ of a_1, a_2 so (5) of Theorem II.2.1 gives the result.

The last statement follows trivially from the Krein-Milman theorem.

In fact this extension result still holds for functions which take values in a compact convex set.

THEOREM II.2.4. If S is a Bauer simplex, and K a compact convex set, and $f : \partial_e S \rightarrow K$ is continuous, then f has an extension to a continuous affine map $\tilde{f} : S \rightarrow K$.

Define $\tilde{f}(s)$ to be that point of K such that $a(\tilde{f}(s)) = \mu_s(a \circ f)$ for all $a \in A(K)$. It is clear that $\tilde{f} : S \rightarrow K$ is affine. If $s \in \partial_e S$, then $\mu_s = \varepsilon_s$ so $\mu_s(a \circ f) = a(f(s))$ for all $a \in A(K)$ so

$\tilde{f}(s) = f(s)$. Finally for each $a \in A(K)$, $s \mapsto a(\tilde{f}(s)) = \mu_s(a \circ f)$ is continuous by II.2.1 (3). It follows that f is continuous.

Faces of Bauer simplexes are easy to describe and have some nice properties.

THEOREM II.2.5. If S is a Bauer simplex, then a subset F of S is a closed face of S if and only if there is a closed subset $D \subset \partial_e S$ such that $F = \overline{\text{co}}(D)$. In this case we have $D = \partial_e F$. In particular a closed face of a Bauer simplex is a Bauer simplex.

If F is a closed face of S , then $\partial_e F = F \cap \partial_e S$ is closed. By the Krein-Milman theorem $F = \overline{\text{co}}(\partial_e F)$. If, on the other hand, $D \subset \partial_e S$ is closed and $0 < \lambda < 1$, $k_1, k_2 \in S$ with $\lambda k_1 + (1 - \lambda)k_2 \in \overline{\text{co}}(D)$, suppose μ_{k_1} is not supported by D . Then $\lambda \mu_{k_1} + (1 - \lambda) \mu_{k_2}$ must assign positive mass to $\partial_e S \setminus D$ as both measures are positive. But the resultant is in $\overline{\text{co}}(D)$ so this is impossible. I.e. $k_1, k_2 \in F$ as claimed. Finally if F is a closed face of S then F is a simplex by I.1.15 and $\partial_e F$ is closed since $\partial_e F = F \cap \partial_e S$.

THEOREM II.2.6. If S is a Bauer simplex and F is a face of S then so is $\overline{F}$.

Let $\mathfrak{F}$ be the set of all $\mu \in P(S)$ with support in $\partial_e S$ and resultant in F . Let $B = \bigcup \{ \text{supp}(\mu) : \mu \in \mathfrak{F} \} \subset \partial_e S$. By II.2.5 it suffices to show that $\overline{F} = \overline{\text{co}}(B)$. Certainly if $k \in F$, μ_k is supported by $\partial_e S$, so $\text{supp}(\mu_k) \subset B$ and $k \in \overline{\text{co}}(B)$. I.e. $F \subset \overline{\text{co}}(B)$ so certainly $\overline{F} \subset \overline{\text{co}}(B)$.

Suppose, conversely, that $k \in F$. If $x \in \text{supp}(\mu_k)$ let V be a closed convex neighbourhood of x with $\mu_k(V) \neq 0$. We show $x \in \overline{F}$, so that $B \subset \overline{F}$ and hence $\overline{\text{co}}(B) \subset \overline{F}$. If $\mu_k(V) = 1$ then $k \in V$ so that $F \cap V \neq \emptyset$. On the other hand if $\mu_k(V) < 1$ let $\mu_1 = \mu_k(V)^{-1}(\mu_k)_V$ and $\mu_2 = (1 - \mu_k(V))^{-1}(\mu_k)_{(S \setminus V)}$. Now

$\mu_k = \mu_k(V)\mu_1 + (1 - \mu_k(V))\mu_2$, so $k = \mu_k(V)r(\mu_1) + (1 - \mu_k(V))r(\mu_2)$. As F is a face of S , $r(\mu_1), r(\mu_2) \in F$. It follows now that again we have $F \cap V \neq \emptyset$. The proof is now complete.

We finish this section by translating Theorems II.1.5 and II.1.8 into analogous theorems for compact Hausdorff spaces.

COROLLARY II.2.7. (Kateřev) Let $\mathcal{N}$ be a compact Hausdorff space, $f : \mathcal{N} \rightarrow [-\infty, \infty)$ be u.s.c. and $g : \mathcal{N} \rightarrow (-\infty, \infty]$ be l.s.c. If $f \leq g$ there is $h \in C(\mathcal{N})$ with $f \leq h \leq g$.

Identify $\mathcal{N}$ with $\partial_e P(\mathcal{N})$. If $\tilde{f} : P(\mathcal{N}) \rightarrow [-\infty, \infty)$ is defined to be equal to f on $\mathcal{N}$ and $-\infty$ on $P(\mathcal{N}) \setminus \mathcal{N}$, then $\tilde{f}$ is u.s.c. and convex. Define similarly $\tilde{g} : P(\mathcal{N}) \rightarrow (-\infty, \infty]$ by $\tilde{g}|_{\mathcal{N}} = g$ and $\tilde{g}|_{P(\mathcal{N}) \setminus \mathcal{N}} = \infty$. If $\tilde{h}$ separating $\tilde{f}$ and $\tilde{g}$ is continuous and affine, then $\tilde{h}|_{\mathcal{N}}$ is the required h .

COROLLARY II.2.8. (Michaels) Let $\mathcal{N}$ be a compact Hausdorff space and E a Frechet space. If $\tilde{\Phi} : \mathcal{N} \rightarrow 2^E$ is l.s.c. and $\tilde{\Phi}(w)$ is a closed convex non-empty subset of E for every $w \in \mathcal{N}$, then there is a continuous selection for $\tilde{\Phi}$.

$\tilde{\Phi} : P(\mathcal{N}) \rightarrow 2^E$ is defined by $\tilde{\Phi}(w) = \tilde{\Phi}(w)$ if $w \in \mathcal{N}$, and $\tilde{\Phi}(w) = E$ if $w \in P(\mathcal{N}) \setminus \mathcal{N}$. $\tilde{\Phi}$ is l.s.c. and affine, any continuous affine selection for $\tilde{\Phi}$ when restricted to $\mathcal{N}$ will satisfy all that is required.

3 Faces.

In this section we shall look at faces of general simplexes. Although we have no precise description of closed faces in general, we do have some useful characterisations of them.

THEOREM II.3.1. Let S be a compact simplex and F a closed convex subset of S . F is a face of S if and only if $\partial_e F \subset \partial_e S$.

The "only if" is obvious. Suppose f is continuous and convex, as well as $\hat{f}$ and B_f we shall utilise $\hat{f}^F = \inf \{a : f|_F \leq a \in A(F)\}$ and $B_f^F = \{k \in F : \hat{f}^F(k) = f(k)\}$. It is clear that $\hat{f}^F \leq \hat{f}|_F$. Suppose $a \in A(S)$ and that $a|_F \geq f|_F$. As S is a simplex, $\hat{f}$ is u.s.c. and affine and coincides with f on $\partial_e S$. Thus $a|_{\partial_e F} \geq \hat{f}|_{\partial_e F}$ and by Corollary I.2.3 $a|_F \geq \hat{f}|_F$. Thus $\hat{f}^F \geq \hat{f}|_F$, and hence $B_f^F = B_f \cap F$ for all f that are continuous and convex.

Suppose F is not a face of S , so that $k \in F$ and $k = (1/2)k_1 + (1/2)k_2$, $k_1, k_2 \in S$, $k_1 \notin F$. Let μ_k be a boundary measure on F representing k , so that μ_k is supported by B_f^F for all continuous convex f , by I.3.8. By the last paragraph, regarded as a measure on S , μ_k will be supported by B_f for all continuous convex f , so is a boundary measure by I.3.8. As S is a simplex, μ_k dominates all measures on S representing k . But let $a \in A(S)$ with $a(k_1) = 1$, $a|_F \leq 0$ and take $f = a \vee 0$. Now $\mu_k(f) = 0 < 1/2 = (1/2)(\xi_{k_1} + \xi_{k_2})(f)$, contradicting the maximality of μ_k .

COROLLARY II.3.2. If G is a closed σ -face of S , or equivalently if G is a closed dilated subset of S , then $\overline{\text{co}}(G)$ is a face of S .

As G is closed, $\partial_e(\overline{\text{co}}(G)) \subset G$ by Milman's theorem. As G is a σ -face, any extreme point of G is an extreme point of S . The result now follows from Theorem II.3.1.

Let us now establish some properties of closed faces of compact simplexes. A face F of S is said to be exposed if there is $a \in A(S)^+$ with $a|_F = 0$ and for all $k \in S \setminus F$ $a(k) > 0$. F is relatively exposed if for all $k \in S \setminus F$ there is $a \in A(S)^+$ with $a|_F = 0$ and $a(k) > 0$.

THEOREM II.3.3. Every closed face F of a simplex S is relatively exposed. F is exposed if and only if F is a G_δ . In particular every closed face of a metrisable simplex is exposed.

Choose $k \in S \setminus F$. By Hahn-Banach separation there is $a \in A(S)$ with $a|_F \leq 0$ and $a(k) > 0$. Define $f = a \vee 0$, which is continuous and convex, and let $g|_F = 0$, $g|_{S \setminus F} = \lambda$, where λ is the supremum of a on S , which is l.s.c. and concave. As $f \leq g$ there is $h \in A(S)$ with $f \leq h \leq g$, by II.1.5. This h has the properties that $h|_F = 0$ and $h(k) \geq a(k) > 0$, so F is relatively exposed.

Clearly if F is exposed, it must be a G_δ . Suppose that C_n are compact subsets of $S \setminus F$ and that $S \setminus F = \bigcup_{n=1}^{\infty} C_n$. A simple compactness argument gives $h_n \in A(S)^+$ with $h_n|_F = 0$ and $h_n|_{C_n} > 0$. We may also suppose $\|h_n\| = 2^{-n}$. The function $\sum_{n=1}^{\infty} h_n$ has the desired properties.

We can establish a characterisation of closed faces in terms of the order structure of $A(S)$. Recall that an ideal was defined on page 9.

THEOREM II.3.4. Let H be an ideal in $A(S)$, S a compact simplex. The set $F = \{k \in S : h(k) = 0 \text{ for all } h \in H\}$ is a closed face of S , and the set $\{a \in A(S) : a|_F = 0\}$ is the closure of H and is also an ideal.

That F is closed and convex is clear. If $k_1, k_2 \in S$, $0 < \lambda < 1$ and $\lambda k_1 + (1 - \lambda)k_2 \in F$, let $h \in H$ with $h(k_1) > 0$ else F is certainly a face. If $h' \in H$, $h' \geq h$, 0 then we have $h'(\lambda k_1 + (1 - \lambda)k_2) \geq \lambda h(k_1) > 0$ - a contradiction.

That $\bar{H} \subset \{a \in A(S) : a|_F = 0\}$ is clear. Let $0 \leq a_0 \notin \bar{H}$, then there is $f \in A(S)^*$ with $f|_H = 0$ and $f(a_0) > 0$. If $h \in H^+$, $f^+(h) = \sup \{f(h') : 0 \leq h' \leq h\} = 0$ as H is an ideal in $A(S)$. Thus $f^+|_H = 0$ as H is positively generated. It follows that f^+ is a positive multiple of evaluation at some point of F . As $f^+(a_0) \geq f(a_0) > 0$, a_0 is not in the set $\{a \in A(S) : a|_F = 0\}$.

If $a \in A(S)$, $a|_F = 0$ we need only find $b \in A(S)^+$ with $b|_F = 0$ and $b \geq a$ to complete the proof. This can be found by using II.1.5 to separate the function that is 0 on F and ∞ on $S \setminus F$ from $a \vee 0$.

We have one other topology that we can define on a simplex. Namely if S is embedded in $A(S)^*$ in the usual way, then S can be given the norm topology of $A(S)^*$. If S is a Bauer simplex this is simply the usual norm of measures. For this topology we can extend the result of II.2.6 to all simplexes.

THEOREM II.3.5. Let F be a face of a simplex S , then the norm closure of F is also a face of S .

We note first that a face G of S is norm closed if and only if $\tilde{G}$ is a norm closed face of $\tilde{S}$, for suppose G is norm closed, $\lambda_n, \mu \geq 0$, $f_n \in F$ and $g \in S$. Suppose $\lambda_n f_n \rightarrow \mu g$, then $|\lambda_n - \mu| = \left| \|\lambda_n f_n\| - \|\mu g\| \right| \leq \|\lambda_n f_n - \mu g\| \rightarrow 0$. Hence $\|\mu f_n - \mu g\| \leq \|\mu f_n - \lambda_n f_n\| + \|\lambda_n f_n - \mu g\| = |\mu - \lambda_n| + \|\lambda_n f_n - \mu g\| \rightarrow 0$. I.e. $\mu g \in \mu \bar{G} = \mu G \subset \tilde{G}$. Thus it will suffice

to prove that the closure of $\tilde{F}$ is a face of $\tilde{S}$.

Let f be in the closure of $\tilde{F}$, and $0 \leq g \leq f$. Choose $f' \in \tilde{F}$ with $\|f - f'\| < \varepsilon/3$. Then we have $0 \leq g + (f' - g) \wedge 0 \leq f'$, g . As $f \geq g$, we see that $(f' - f) \leq (f' - g)$, thence that $(f' - f) \wedge 0 \leq (f' - g) \wedge 0 \leq 0$.

If $a \in A(S)$ then $a = \|a\| 1_S - (\|a\| 1_S - a)$, so we can write $a = b - c$ with $b, c \in A(S)^+$ and $\|b\| + \|c\| \leq 3\|a\|$. Thus if $h \in A(S)^*$ we see that

$$\begin{aligned} \|(h \wedge 0)\| &= \sup \{ |(h \wedge 0)(a)| : \|a\| \leq 1 \} \\ &\leq 3 \sup \{ |(h \wedge 0)(a)| : \|a\| \leq 1, a \geq 0 \} \\ &= 3 \sup \{ \inf \{ h(b) : 0 \leq b \leq a \} : \|a\| \leq 1, a \geq 0 \} \\ &\leq 3\|h\|, \end{aligned}$$

as $0 \leq b \leq a$ implies $\|b\| \leq \|a\|$.

Thus $\|(f' - f) \wedge 0\| \leq 3\|f' - f\| \leq 3(\varepsilon/3) = \varepsilon$. As the norm is additive on S (and hence on $-S$), we see that $\|(f' - g) \wedge 0\| \leq \varepsilon$. As we have seen that $0 \leq g + (f' - g) \wedge 0 \leq f' \in \tilde{F}$ it follows that $g + (f' - g) \wedge 0 \in \tilde{F}$, so g belongs to the norm closure of F and the proof is complete.

We introduce now a notion that will achieve great importance later. If F is a subset of a compact convex set K , we define the complementary set F' to be the union of all the faces of K that do not meet F . If F' is convex then it is readily seen to be a face, in which case it is termed the complementary face of F . F is a split face if either it is $\emptyset$ or K (in which case it is termed an improper split face) or if F' is convex and each $k \in K \setminus (F \cup F')$ can be written uniquely as a convex combination $k = \lambda x + (1 - \lambda)y$, with $x \in F$, $y \in F'$ and $0 < \lambda < 1$. It is clear that if F is a split face then so is F' .

THEOREM II.3.6. If F is a split face of a compact convex set then F is norm closed.

Embed K in $A(K)^*$ in the usual way. The function, h , taking $k = \lambda x + (1 - \lambda)y$ ($x \in F$, $y \in F'$, $0 < \lambda < 1$) to λ , and that is 1 on F , 0 on F' extends to a linear functional on $A(K)^*$ that is bounded on the unit ball, and is hence norm continuous. Now $F = K \cap h^{-1}(1)$ is norm closed.

For simplexes the converse of this result is also true.

THEOREM II.3.7. Let F be a face of a compact simplex. F is split if and only if F is norm closed.

The proof of II.3.5 shows that if F is norm closed then so is $\tilde{F}$. We identify the simplex, S , with the state space of $A(S)$. Fix $k \in S$, and let $k_1 = \sup \{m \in \tilde{F} : m \leq k\}$ which exists as $S - S = A(S)^*$ is a complete vector lattice. If $m \in \tilde{F}$, let $m = \lambda_m k_m$, $0 \leq \lambda_m$, $k_m \in F$. If $m \leq k$ then $\lambda_m \leq 1$, for $k - m = \mu n$ with $\mu \geq 0$ and $n \in S$. Now we may write

$$k = m + \mu n = (\lambda_m + \mu) \left(\frac{\lambda_m k_m + \mu n}{\lambda_m + \mu} \right)$$

where $\lambda_m + \mu = 1$ by the uniqueness property of the definition of a base. Thus we may define $\alpha = \sup \{\lambda_m : m \leq k\}$, and choose m_0 with $\lambda_{m_0} \geq \alpha - \varepsilon$. For any $m \in \tilde{F}$ with $m_0 \leq m \leq k$, $m - m_0 = \sum c_m$ where $c_m \in F$ and $\sum c_m = \lambda_m - \lambda_{m_0} \leq \varepsilon$. I.e. $\|m - m_0\| \leq \varepsilon$ if $m_0 \leq m \leq k$ and $m \in \tilde{F}$. Thus the net $\{m\}$ is Cauchy and converges to a limit in F . This is clearly the supremum k_1 .

$(k - k_1) \wedge k_1 = 0$, for if $0 \leq x \leq k - k_1$, k_1 then $0 \leq x \leq k_1 + x \leq k$, contradicting the definition of k_1 if $x \neq 0$. I.e. the face generated by $k - k_1$ does not meet $\tilde{F}$, for if $0 \leq x \leq k - k_1$, $x \in \tilde{F}$ then $x \leq k$ as $k_1 \geq 0$, so $x \leq k_1$ and $x = 0$.

We now have a decomposition of the required form:

$$k = \|k_1\| \left(\frac{k_1}{\|k_1\|} \right) + \|k - k_1\| \left(\frac{k - k_1}{\|k - k_1\|} \right).$$

It remains to prove the uniqueness of such a decomposition.

Suppose $k = \lambda m + (1 - \lambda)n$, $m \in F$, $n \in F'$, then clearly $\lambda m \leq k_1$,

so $k - k_1 \leq (1 - \lambda)n$. We have then

$$(1 - \lambda)n \geq (1 - \lambda)n - (k - k_1) \geq 0$$

so $(1 - \lambda)n - (k - k_1) \in \tilde{F}'$. But also $k_1 \geq k_1 - \lambda m =$

$(k - \lambda m) - (k - k_1) = (1 - \lambda)n - (k - k_1) \geq 0$, so we have

$\tilde{F}' \cap \tilde{F} \neq \{0\}$ unless $\mu m = k_1$, thus proving the result.

4 Real valued continuous affine functions.

In this section we deal with real valued continuous affine functions defined on a compact simplex. The results of section 1 provide our tools. We prove a variety of extension theorems, and conclude by proving that $A(S)$ has the approximation property.

If G is a σ -face of S then a function defined on G will be termed affine if its restriction to every face contained in G is affine.

THEOREM II.4.1. Let S be a compact simplex and G a closed σ -face of S . Suppose $f : S \rightarrow [-\infty, \infty)$ is u.s.c. and convex, $g : S \rightarrow (-\infty, \infty]$ is l.s.c. and concave, $h : G \rightarrow \mathbb{R}$ is continuous and affine. If $f|_G \leq h \leq g|_G$ then there is $\bar{h} \in A(S)$ such that $\bar{h}|_G = h$ and $f \leq \bar{h} \leq g$.

Define $\bar{f}(k) = f(k)$ if $k \in S \setminus G$,
 $= h(k)$ if $k \in G$.

We show $\bar{f}$ is u.s.c. and convex. To see that $\bar{f}$ is convex it will suffice to consider $k_1 \in S \setminus G$, $k_2 \in G$ and $0 < \lambda < 1$, for if both elements lie in either G or $S \setminus G$ then the desired inequality is an immediate consequence of our assumptions. As G is a σ -face, $\lambda k_1 + (1 - \lambda)k_2 \in S \setminus G$, so that

$$\begin{aligned} \bar{f}(\lambda k_1 + (1 - \lambda)k_2) &= f(\lambda k_1 + (1 - \lambda)k_2) \\ &\leq \lambda f(k_1) + (1 - \lambda)f(k_2) \text{ (as } f \text{ is convex)} \\ &\leq \lambda f(k_1) + (1 - \lambda)h(k_2) \\ &= \lambda \bar{f}(k_1) + (1 - \lambda)\bar{f}(k_2). \end{aligned}$$

Suppose $\alpha \in \mathbb{R}$, consider

$$\begin{aligned} \{k \in S : \bar{f}(k) \geq \alpha\} &= \{k \in S \setminus G : f(k) \geq \alpha\} \cup \{k \in G : h(k) \geq \alpha\} \\ &= \{k \in S : f(k) \geq \alpha\} \cup \{k \in G : h(k) \geq \alpha\} \end{aligned}$$

(as $h \gg f|_G$) which is closed because f is u.s.c., h is continuous and G is closed.

Define similarly $\bar{g}(k) = g(k)$ if $k \in S \setminus G$,
 $= h(k)$ if $k \in G$.

It may be shown in a similar manner that $\bar{g}$ is l.s.c. and concave. Now choose $\bar{h} \in A(S)$ with $\bar{f} \leq \bar{h} \leq \bar{g}$, using Theorem II.1.5, then this will have the desired properties.

COROLLARY II.4.2. Let S be a compact simplex and F a closed face of it, f_1, f_2, g_1 and $g_2 \in A(S)$ and $h \in A(F)$. If $f_1|_F, f_2|_F \leq h \leq g_1|_F, g_2|_F$ then there is $\bar{h} \in A(S)$ such that $\bar{h}|_F = h$ and $f_1, f_2 \leq \bar{h} \leq g_1, g_2$.

Take $f = f_1 \vee f_2$ and $g = g_1 \wedge g_2$ in II.4.1.

COROLLARY II.4.3. Let S be a compact simplex, F a closed face of it, $h \in A(F)$ and $p : S \rightarrow \mathbb{R}^+$ l.s.c. and concave. If $|h(k)| \leq p(k)$ for all $k \in F$ then there is $\bar{h} \in A(S)$ with $\bar{h}|_F = h$ and $|\bar{h}(k)| \leq p(k)$ for all $k \in S$.

Take $f = p$ and $g = -p$ in II.4.1.

An immediate consequence of this is that any continuous affine function from a closed face of a compact simplex into the reals has a norm preserving extension to the whole of the simplex.

COROLLARY II.4.4. Let K be a compact convex set. K is a simplex if and only if $A(K)$ has the R.D.P.

If K is a simplex then II.4.2 tells us that $A(K)$ has the R.S.P. and thence the R.D.P. Conversely if $A(K)$ has the R.D.P. Lemma I.1.13 tells us that K is a simplex.

Another corollary of II.4.1 is presented now. The interest here is in the converse which is valid if K is metrisable.

THEOREM II.4.5. Let K be a compact convex set. Consider the property:

(E) For every compact subset D of $\partial_e K$ and every $h \in C(D)$ there is $\bar{h}$ with $\bar{h}|_D = h$ and $\|\bar{h}\| = \|h\|$.

Every simplex has this property. If K is metrisable this is equivalent to K being a simplex.

That simplexes have this property follows from II.4.1 with $f = -\|h\|$, $g = \|h\|$. Assume now that K is metrisable and has property (E), we show that K is a simplex.

Let $k \in K$ and let μ, ν be two maximal probability measures representing k . We show that $\mu = \nu$, which will prove that K is a simplex. As K is metrisable $\partial_e K$ is a G_δ and $\mu(\partial_e K) = \nu(\partial_e K) = 1$. By regularity, for any $\varepsilon > 0$ there is a compact subset D of $\partial_e K$ such that

$$\mu(D) > 1 - \varepsilon, \quad \nu(D) > 1 - \varepsilon.$$

If $g \in C(K)$ there is $a \in A(K)$ with $a|_D = g|_D$ and $\|a\| \leq \|g\|$.

Now we have:

$$\begin{aligned} |\mu(g) - \nu(g)| &\leq \left| \int_D g d\mu - \int_D g d\nu \right| + \left| \int_{K \setminus D} g d\mu - \int_{K \setminus D} g d\nu \right| \\ &\leq \left| \int_D a d\mu - \int_D a d\nu \right| + \left| \int_{K \setminus D} g d\mu \right| + \left| \int_{K \setminus D} g d\nu \right| \\ &\leq \left| \int_K a d\mu - \int_K a d\nu \right| + \left| \int_{K \setminus D} a d\mu \right| + \left| \int_{K \setminus D} a d\nu \right| \\ &\quad + \left| \int_{K \setminus D} g d\mu \right| + \left| \int_{K \setminus D} g d\nu \right| \\ &\leq 4 \varepsilon \|g\|. \end{aligned}$$

As this holds for all $\varepsilon > 0$, $\mu(g) = \nu(g)$ for each $g \in C(K)$.

Hence $\mu = \nu$ and the proof is complete.

The assumption that K be metrisable for the last equivalence in this result is essential. In fact Alfsen has given an example of a compact convex set with property (E) which is not a simplex.

We have looked so far only at extensions of single functions. We now prove an extension of a theorem of Borsuk-Dugundji-Arens.

THEOREM II.4.6. Let F be a closed metrisable face of a compact simplex S . There is a linear operator $T : A(F) \longrightarrow A(S)$ such that:

- (1) $\underline{Ta|_F = a} \quad (a \in A(F))$
- (2) $\underline{a \geq 0 \Rightarrow Ta \geq 0} \quad (a \in A(F))$
- (3) $\underline{T1_F = 1_K}$
- (4) $\underline{\|Ta\| = \|a\|} \quad (a \in A(F)).$

If we construct a continuous affine map $\psi : S \longrightarrow F$ such that $\psi|_F$ is the identity, and define $(Ta)(k) = a(\psi k)$, then it easily verified that T has the desired properties.

Let $\{f_n\}_{n=1}^{\infty}$ be a dense subset of $A(F)$, which exists as F is metrisable (I.1.17), and let τ be the locally convex topology induced on $A(F)^*$ by $\{f_n\}_{n=1}^{\infty}$. The natural injection of F into $A(F)^*$ is a homeomorphism for the given topology of F and the τ -topology on $A(F)^*$. The topology τ is metrisable as it is defined by a countable family of semi-norms. Let E be the τ -completion of $A(F)^*$, so that E is a Frechet space. We identify F with its canonical copy in E , which is compact and hence closed. Now define a map $\bar{\Phi} : S \longrightarrow 2^E$ by

$$\begin{aligned} \bar{\Phi}(k) &= F \quad (k \in S \setminus F) \\ &= \{k\} \quad (k \in F). \end{aligned}$$

That $\bar{\Phi}$ is affine is clear. That it is lower semi-continuous is because if $U \subset E$ is open, then $\{k \in S : \bar{\Phi}(k) \cap U \neq \emptyset\}$ is either empty (if $U \cap F = \emptyset$) or is else $(S \setminus F) \cup (F \cap U)$. As the τ -topology on F coincides with that given, there is $V \subset S$ which is open and with $V \cap F = U \cap F$. Now $(S \setminus F) \cup (F \cap U) = (S \setminus F) \cup V$ is open as required. Now any continuous affine selection for

$\bar{\Phi}$ will suffice to complete the proof, such existing because of II.1.8.

We shall conclude this section by showing that $A(S)$ has the approximation property. I.e. if $C \subset A(S)$ is a compact set, and $\varepsilon > 0$, then there is a bounded linear operator $T_\varepsilon : A(S) \rightarrow A(S)$, such that $\dim(T_\varepsilon(A(S))) < \infty$ and for all $c \in C$ we have $\|T_\varepsilon - c\| \leq \varepsilon$.

Before proceeding with the proof of this we need a result on the extension of compact operators. We remind the reader at this point of Theorem I.1.19.

PROPOSITION II.4.7. Let $X \subset Y$ be Banach spaces and S a compact simplex. If T is a compact linear operator from X into $A(S)$ and $\varepsilon > 0$, then there is a compact extension $\bar{T}$ of T taking Y into $A(S)$ with $\|\bar{T}\| \leq (1 + \varepsilon) \|T\|$.

Let B denote the unit ball of X^* . We define

$\Phi : \|T\| B \rightarrow 2^{Y^*}$ by

$$\Phi(f) = \{g \in Y^* : g|_X = f, \|g\| \leq (1 + \varepsilon) \|T\|\}.$$

Obviously Φ is affine and $\Phi(f)$ is closed for every $f \in \|T\| B$.

Let $U \subset Y^*$ be open and choose $f_0 \in \{f \in \|T\| B : \Phi(f) \cap U \neq \emptyset\}$ (if none such exists we have nothing to prove). As $\|f_0\| \leq \|T\|$ and U is open, there is $g_0 \in \Phi(f_0) \cap U$ with $\|g_0\| < (1 + \varepsilon) \|T\|$.

Let V be an open neighbourhood of g_0 contained in

$U \cap \{g \in Y^* : \|g\| < (1 + \varepsilon) \|T\|\}$. Since the restriction map

from Y^* onto X^* is open, $\{h|_X : h \in U\}$ is open in X^* . Clearly

$f_0 \in \{h|_X : h \in U\} \cap \|T\| B \subset \{f \in \|T\| B : \Phi(f) \cap U \neq \emptyset\}$, so

this latter set is relatively open in $\|T\| B$. We have thus

shown that Φ is lower semi-continuous.

Now let $\pi : S \rightarrow \|T\| B$ be such that $(Tx)(k) = (\pi k)(x)$

($x \in X, k \in S$). The function $\bar{\Phi} \circ \pi : S \rightarrow 2^{Y^*}$ clearly satisfies

all the conditions of Theorem I.1.8, so has a continuous affine selection $\psi : S \longrightarrow \{g \in Y^* : \|g\| \leq (1 + \varepsilon) \|T\|\}$. The operator $\bar{T}$ defined by $(\bar{T}y)(k) = (\psi k)(y)$ ($y \in Y, k \in S$) now has all the desired properties.

THEOREM II.4.8. Let S be a compact simplex, then $A(S)$ has the approximation property.

Suppose $B \subset A(S)$ is compact. Choose $1 > \varepsilon > 0$ and let $\{x_1, x_2, \dots, x_n\}$ be an ε -net for B , with linear span H . If h belongs to the unit sphere in H , choose $f_h \in H^*$ with $|f_h(h)| = 1$, $\|f_h\| = 1$, and let $U_h = \{h' \in H : \|h'\| = 1, |f_h(h')| > 1 - \varepsilon\}$. By compactness, a finite number of these open sets cover the unit sphere in H . Let the corresponding linear functionals be $\{f_1, f_2, \dots, f_m\}$. Now let l_m^∞ be the m -dimensional space with the l^∞ -norm. Define a map $V : H \longrightarrow l_m^\infty$ by $Vh = (f_1(h), \dots, f_m(h))$, so that $\|h\| \geq \|Vh\| \geq (1 - \varepsilon) \|h\|$ for all $h \in H$. I.e. $1 \geq \|V\| \geq 1 - \varepsilon$, so in particular V is $(1, 1)$. Let V^{-1} denote the inverse of V mapping $V(H)$ onto H with $1 \leq \|V^{-1}\| \leq (1 - \varepsilon)^{-1}$. By II.4.7 V^{-1} has an extension T mapping l_m^∞ into $A(S)$ with $\|T\| \leq (1 + \varepsilon)/(1 - \varepsilon)$. On the other hand by the Hahn-Banach theorem there is a norm preserving extension U of V taking $A(S)$ into l_m^∞ . Now consider $T \circ U$ which has norm at most $(1 + \varepsilon)/(1 - \varepsilon)$, certainly has finite dimensional range, and is the identity on H . If $b \in B$ and $\|x_1 - b\| \leq \varepsilon$ then

$$\begin{aligned} \|(T \circ U)(b) - b\| &\leq \|(T \circ U)(b) - (T \circ U)(x_1)\| + \|(T \circ U)(x_1) - b\| \\ &\leq \|T \circ U\| \|b - x_1\| + \|x_1 - b\| \\ &\leq \varepsilon((1 + \varepsilon)/(1 - \varepsilon) + 1), \end{aligned}$$

which can be made as small as desired. This completes the proof.

5 Banach space valued continuous affine functions.

In this section we shall look at continuous affine functions on a compact simplex, with values in a Banach space. There are two distinct cases to consider, the first being when the Banach space involved is ordered. In such cases there are three types of property that interest us. The first concerns the normality of the spaces of functions, such results being trivial. The second concerns the positive generation of the space, and the third is various forms of dominated extension theorem. In the non-ordered case, only the last type of result has any meaning.

E will denote a Banach space. If K is a compact convex set, then $A(K, E)$ will denote the space of all continuous affine E -valued functions on K . If $a \in A(K, E)$ we define $\|a\| = \sup \{ \|a(k)\| : k \in K \}$, for which norm $A(K, E)$ is a Banach space. We call E an ordered Banach space if it possesses a closed cone, P . In this case we give $A(K, E)$ the cone $\{ a \in A(K, E) : a(k) \in P \text{ for all } k \in K \}$. B_α will denote the open ball in E of radius α and centre the origin, $P_\alpha = P + B_\alpha$.

We look first at the ordered case. E is said to satisfy $G(\alpha)$ ($\alpha \geq 1$) if $x \in X$ and $x \in B_1$ implies there exists $y \in B_\alpha$ with $y \geq x, -x$.

THEOREM II.5.1. Let E be an ordered Banach space, S a compact simplex and $\alpha \geq 1$. E satisfies $G(\alpha)$ if and only if $A(S, E)$ satisfies $G(\alpha)$.

Suppose E satisfies $G(\alpha)$ and $a \in A(S, E)$. It will suffice to show that if $\|a\| \leq 1$ and $\epsilon > 0$ there is $b \in A(S, E)$ with

$b \geq a$, $-a$ and $\|b\| \leq \alpha + \varepsilon$. Define $B : S \rightarrow 2^E$ by

$$B(k) = \{e \in E : e \geq a(k), -a(k), \|e\| < \alpha + \varepsilon\}^-.$$

It is clear that $B(k)$ is closed and convex for all $k \in S$, and is non-empty as E satisfies $G(\alpha)$. If we show that B is l.s.c. and affine, then choosing a continuous affine selection for B will provide the desired b .

Suppose $k, k' \in S$ and $0 < \lambda < 1$. Let $e \in B(k)$, $e' \in B(k')$, we must show that $\lambda e + (1 - \lambda)e' \in B(\lambda k + (1 - \lambda)k')$. Choose $e_n \in B(k)$ (resp. $e'_n \in B(k')$) with $\|e_n\| < \alpha + \varepsilon$ (resp. $\|e'_n\| < \alpha + \varepsilon$) and $\|e_n - e\| \leq n^{-1}$ (resp. $\|e'_n - e'\| \leq n^{-1}$). It is clear that $\lambda e_n + (1 - \lambda)e'_n \in a(\lambda k + (1 - \lambda)k')$ and that

$$\|\lambda e_n + (1 - \lambda)e'_n\| \leq \lambda \|e_n\| + (1 - \lambda) \|e'_n\| < \alpha + \varepsilon.$$

As also

$$\|(\lambda e_n + (1 - \lambda)e'_n) - (\lambda e + (1 - \lambda)e')\| \leq \|\lambda(e - e_n)\| + \|(1 - \lambda)(e' - e'_n)\| \leq n^{-1},$$

we must have $\lambda e + (1 - \lambda)e' \in B(\lambda k + (1 - \lambda)k')$.

Now suppose $D \subset E$ is open, $k_0 \in S$ and that $B(k_0) \cap D \neq \emptyset$. Clearly we must be able to find $d_0 \in B(k_0) \cap D$ with $\|d_0\| < \alpha + \varepsilon$. Let $\|d_0\| = (\alpha + \varepsilon) - \delta$, and suppose $\{d : \|d - d_0\| < \eta\} \subset D$. Let U be the open set $\{k \in K : \|a(k) - a(k_0)\| < \min\{\eta, \delta\}/2\}$. By assumption there is $f \in E$ with $f \geq \pm(a(k) - a(k_0))$ and $\|f\| < (1/2)\min\{\eta, \delta\}$ if $k \in U$. Now $d = d_0 + f \geq \pm a(k)$ and $\|d\| \leq \|d_0\| + \|f\| \leq \alpha + \varepsilon - (\delta/2)$ so $d \in B(k)$. But also $\|d - d_0\| < \eta/2$ so $d \in D$. Now if $k \in U$, $B(k) \cap D \neq \emptyset$, so that B is l.s.c.

The converse is obvious.

We first partially generalise Theorem II.4.1. Note first that in a general ordered Banach space the notion of lower (or upper) semi-continuity seems to be of little use. We accordingly restrict our attention to continuous functions.

THEOREM II.5.2 Let E be an ordered Banach space, S a compact simplex, and F a closed face of S . Suppose $f : S \rightarrow E$ is continuous and convex and $g : S \rightarrow E$ is continuous and concave. If $h \in A(F, E)$ and $f|_F \leq h \leq g|_F$ then there is $\bar{h} \in A(S, E)$ with $\bar{h}|_F = h$ and $f \leq \bar{h} \leq g$.

$$\begin{aligned} \text{If } Q : S \rightarrow 2^E \text{ is defined by } Q(k) &= \{h(k)\} \quad (k \in F) \\ &= E \quad (k \in S \setminus F), \end{aligned}$$

then Q is l.s.c. and affine. The maps $k \mapsto \{f(k)\}$ and $k \mapsto \{g(k)\}$ are l.s.c., so by the proof of Lemma II.1.6

$$H_\alpha(k) = Q(k) \wedge (f(k) + P_\alpha) \wedge (g(k) - P_\alpha)$$

defines a l.s.c. map from S into 2^E . It is clear that H_α is affine. If

$$H(k) = Q(k) \wedge (f(k) + P) \wedge (g(k) - P)$$

then $H(k)$ is closed and affine. If we note that $H \subset H_\alpha \subset H + B_\alpha$ for all $\alpha > 0$, we may apply II.1.10 to obtain a continuous affine selection for H , which will satisfy all the required conditions.

We can similarly generalise II.4.2, although we need slightly more assumptions on E . Note however that this does not follow from the last result in the same way as we cannot take a "pointwise supremum" in E . First we prove a lemma. Let

$$A(a, b, c, d) = (a + P) \wedge (b + P) \wedge (c - P) \wedge (d - P)$$

$$A_\alpha(a, b, c, d) = (a + P_\alpha) \wedge (b + P_\alpha) \wedge (c - P_\alpha) \wedge (d - P_\alpha).$$

LEMMA II.5.3. If E is positively generated, has the R.S.P. and P is normal there exists $\beta > 0$ such that for any $a, b \leq c, d$,

$$\underline{A_\alpha(a, b, c, d) \subset A(a, b, c, d) + B_{\alpha\beta}}.$$

If $x \in A_\alpha(a, b, c, d)$ then

$$a - z_1, b - z_2 \leq x \leq c + z_3, d + z_4 \quad (\|z_i\| < \alpha).$$

For some constant $\gamma > 0$ (depending only on E) there is

$z \geq z_1, z_2, z_3, z_4, 0$ with $\|z\| < \gamma\alpha$. Hence $a, b, x - z \leq c, d, x + z$.

As E has the R.S.P. there is y such that

$$a, b, x - z \leq y \leq c, d, x + z,$$

so that $y \in A(a, b, c, d)$ and since P is δ -normal for some $\delta > 0$,

$\|y - x\| \leq \delta \|z\| \leq \delta\gamma\alpha$. If we take $\beta = \gamma\delta$ the result is proved.

THEOREM II.5.4. Let E be an ordered Banach space with the R.D.P. and with P generating and normal. Suppose S is a compact simplex and F a closed face of S . Let f_1, f_2, g_1 and g_2 lie in $A(S, E)$ and $h \in A(F, E)$ be such that $f_1|_F, f_2|_F \leq h \leq g_1|_F, g_2|_F$. There is $\bar{h} \in A(S, E)$ such that $\bar{h}|_F = h$ and $f_1, f_2 \leq \bar{h} \leq g_1, g_2$.

$$\begin{aligned} \text{If } Q : S \rightarrow 2^E \text{ is defined by } Q(k) &= \{h(k)\} \quad (k \in F) \\ &= E \quad (k \in S \setminus F) \end{aligned}$$

then Q is affine, l.s.c. and $Q(k)$ is closed for each $k \in S$. The assumptions of the theorem tell us that

$$H(k) = Q(k) \cap (f_1(k) + P) \cap (f_2(k) + P) \cap (g_1(k) - P) \cap (g_2(k) - P)$$

is non-empty for each $k \in S$. Clearly H is affine. If we also define

$$H_\alpha(k) = Q(k) \cap (f_1(k) + P_\alpha) \cap (f_2(k) + P_\alpha) \cap (g_1(k) - P_\alpha) \cap (g_2(k) - P_\alpha)$$

then lemma II.1.6 shows that H_α is l.s.c. and affine. Now

lemma II.5.3 tells us that $H \subset H_\alpha \subset H + B_{\alpha\beta}$ for all $\alpha > 0$. We may now apply I.1.10 to obtain h .

We may also enquire when $A(S, E)$ is a lattice. It is easily verified that for this to occur E and $A(S)$ must be lattices, so that we are dealing simply with $C(\partial_e S, E)$. Now if we assume that P is normal and that E is a lattice then if $f \in C(\partial_e S, E)$ the function $f^+ : k \mapsto f(k)^+$ is easily verified to be continuous, so that $C(\partial_e S, E)$ is a lattice.

We may combine II.5.1 and II.5.4 as follows. We recall that an ordered Banach space E is termed a Banach lattice if it is a lattice and $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. If E is an ordered Banach space E^* may be ordered by the wedge $\{f \in E^* : f(x) \geq 0 \text{ for all } x \in P\}$. Davies has shown that E^* is a Banach lattice if and only if E has the R.S.P. and E also satisfies both $G(1)$ and $N(1)$ where $N(\alpha)$ is the condition:

$$-x \leq y \leq x \implies \|y\| \leq \alpha \|x\|.$$

The following corollary is easy to verify.

COROLLARY II.5.5. Let E be an ordered Banach space and S a compact simplex. E^* is a Banach lattice if and only if $A(S, E)^*$ is a Banach lattice.

Before we leave ordered Banach spaces completely, we present a simple result on the existence of positive extensions.

PROPOSITION II.5.6. Let S be a compact simplex, F a closed face of S and E a Banach space. If $h \in A(F, E)$ there is $\bar{h} \in A(S, E)$ with $\bar{h}|_F = h$ and $\bar{h}(S) = h(F)$.

$$\begin{aligned} \text{Take } H(k) &= \{h(k)\} \quad (k \in F) \\ &= h(F) \quad (k \in S \setminus F). \end{aligned}$$

This satisfies I.1.8 and any continuous affine selection of H will satisfy the requirements.

COROLLARY II.5.7. Let S be a compact simplex, F a closed face of S and E an ordered Banach space. If $h \in A(F, E)$ and $h \geq 0$ there is $\bar{h} \in A(S, E)$ such that $\bar{h}|_F = h$, $\bar{h} \geq 0$ and $\|\bar{h}\| = \|h\|$.

We conclude by generalising II.4.3 to the case of functions with values in a (non-ordered) Banach space.

THEOREM II.5.8. Let S be a compact simplex, F a closed face of S and E a Banach space. Let $p : S \rightarrow \mathbb{R}^+ \setminus \{0\}$ be l.s.c. and concave, and suppose $h \in A(F, E)$ with $\|h(k)\| \leq p(k)$ for all $k \in F$. There is $\bar{h} \in A(S, E)$ such that $\bar{h}|_F = h$ and $\|\bar{h}(k)\| \leq p(k)$ for all $k \in S$.

$$\begin{aligned} \text{Define } Q(k) &= \{h(k)\} \quad (k \in F) \\ &= E \quad (k \in S \setminus F) \end{aligned}$$

so that Q is l.s.c. and affine. The map $k \mapsto \{e \in E : \|e\| < p(k)\}$ is l.s.c. (as p is l.s.c.) and affine (as p is concave and strictly positive). If we let

$$H(k) = Q(k) \cap \{e \in E : \|e\| \leq p(k)\}$$

then $H(k)$ is closed and H is affine. Now define

$$H_\alpha(k) = Q(k) \cap (\{e \in E : \|e\| < p(k)\} + B_\alpha)$$

which by II.1.6 is l.s.c. and affine. Noting that $H \subset H_\alpha \subset H + B_\alpha$ for all $\alpha > 0$ we may apply I.1.8 to obtain a continuous affine selection for H which satisfies all that is required of it.

CHAPTER III

GENERAL A(K) SPACES

1 Faces and order ideals.

Before the main material of this section we devote a little attention to the formation of norms in ordered vector spaces. We have already defined the notion of a base B for the positive cone in E . If E is positively generated then $\text{co}(B \cup -B)$ is an absorbent convex set. Its gauge is the base-seminorm induced by B , $\| \cdot \|_B^B$. If there is a Hausdorff vector topology τ on E for which B is compact then $\| \cdot \|_B^B$ is a norm and E is $\| \cdot \|_B^B$ -complete. To see that it is a norm suppose $\| x \|_B^B = 0$ then for all $n \in \mathbb{N}$ there are $0 \leq \alpha_n \leq 1$, $a_n, b_n \in B$ with $nx = \alpha_n a_n + (\alpha_n - 1)b_n$. By choosing an appropriate subnet we may suppose $\alpha_n \rightarrow \alpha$, $a_n \rightarrow a \in B$ and $b_n \rightarrow b \in B$ (τ -convergence). Now $n'x \rightarrow \alpha a + (\alpha - 1)b$ (τ -convergence) which is only possible if $x = 0$. To see that E is $\| \cdot \|_B^B$ -complete suppose (q_n) is a Cauchy sequence in E . For each $\xi > 0$ there is $N(\xi)$ such that $n \geq m \geq N(\xi)$ implies that $q_n \in q_m + \xi \text{co}(B \cup -B)$. The sequence (q_n) is $\| \cdot \|_B^B$ -bounded, so there is μ with $q_n \in \mu \text{co}(B \cup -B)$ for all n . Let $q_{n'}$ be a subnet of (q_n) τ -converging to q in $\mu \text{co}(B \cup -B)$, which exists as $\mu \text{co}(B \cup -B)$ is τ -compact. As $\xi \text{co}(B \cup -B)$ is closed, $q \in q_m + \xi \text{co}(B \cup -B)$ whenever $m \geq N(\xi)$. I.e. $q_m \rightarrow q$ for $\| \cdot \|_B^B$. Corollary I.1.11 shows that the norm in $A(K)^*$ is the base norm induced by $S(A(K))$.

$e \in E$ is an order unit if for each $x \in E$ there is $\lambda \geq 0$ with $-\lambda e \leq x \leq \lambda e$. If e is an order unit then $[-e, e] = \{x \in E : -e \leq x \leq e\}$ is absorbent. Its gauge is a semi-norm on E . If E is Archimedean then this is a norm, for if $\lambda x \in [-e, e]$ for all $\lambda > 0$ then $-\xi e \leq x \leq \xi e$ for all $\xi > 0$ and $x = 0$. This is the order unit norm induced by e , $\| \cdot \|_e$. Let $S(E) =$

$\{ f \in E^* : f(e) = \|f\| = 1 \}$, the state space of E , and map $E \longrightarrow A(S(E))$ by $x \longmapsto \hat{x}$ where $\hat{x}(f) = f(x)$. $x \longmapsto \hat{x}$ is a positive map and is clearly norm-decreasing. In fact it is an isometry for if $x \in E$ and $\|x\|_e = 1$, consider the linear span of x and e (see Figure 1). Clearly there is a linear functional q on this span with $\|q\| = |q(x)| = q(e) = 1$. Extend q with preservation of norm to $\bar{q}$ on E . Now $\|\hat{x}\| \geq |\hat{x}(\bar{q})| = |\bar{q}(x)| = |q(x)| = 1$. If E is $\|\cdot\|_e$ -complete then E is isometric to $A(S(E))$. This is an order isomorphism also for if $\|x\|_e \leq 1$ then $x \geq 0$ if and only if $\|x - e\|_e \leq 1$ if and only if $\|\hat{x} - \hat{e}\| \leq 1$ if and only if $\hat{x} \geq 0$. Thus if $x \in E$, $x \geq 0$ if and only if $p(x) \geq 0$ for all $p \in S(E)$.

An order ideal in an ordered vector space is a linear subspace J such that $0 \leq x \leq y \in J$ implies $x \in J$. Thus being an order ideal is a weaker property than being an ideal. In this section we look at the relationship between faces of K , order ideals in $A(K)$ and quotients of $A(K)$ by such ideals. Throughout this section it will be convenient to consider K embedded in $A(K)^* = E$ in the usual manner. If $a \in A(K)$ we may also regard a as a (weak*) continuous linear functional on E , this we do with no further explanation. E will be ordered by the usual cone, and will normally be considered with the weak* topology as the dual of $A(K)$, but at times also with the dual norm topology.

We now make some definitions. If $G \subset K$ we write

$$G^\perp = \{ a \in A(K) : a|_G = 0 \},$$

and if $B \subset A(K)$,

$$B^\perp = \{ k \in K : a(k) = 0 \text{ for all } a \in B \}$$

$$\text{and } B^0 = \{ x \in E : a(x) = 0 \text{ for all } a \in B \}.$$

The next result is a straightforward verification.

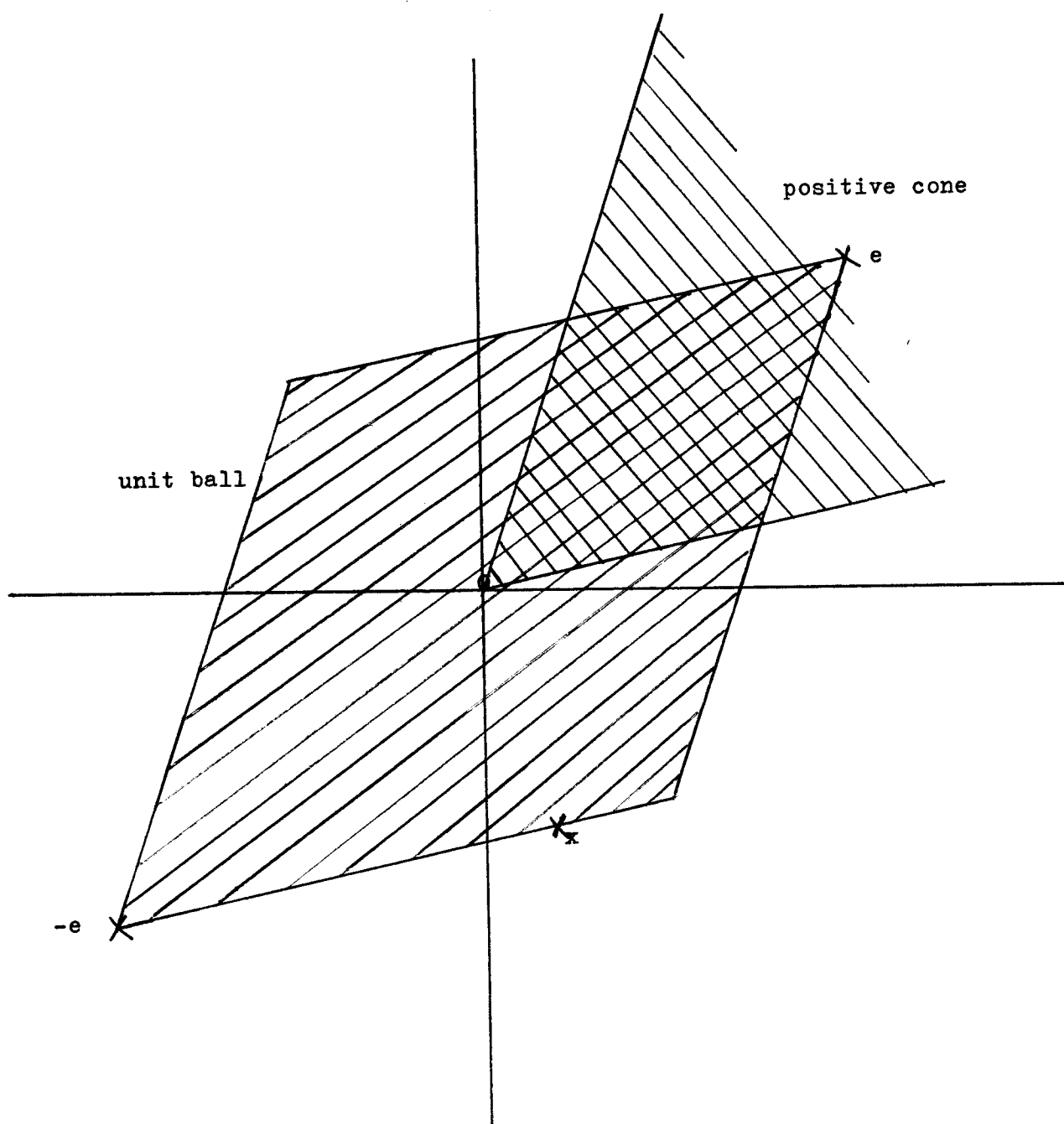


FIGURE 1

PROPOSITION III.1.1. If $G \subset K$, $G^\perp$ is a closed order ideal in $A(K)$. If $B \subset A(K)$ then $B^\perp$ is a closed convex subset of K . If also every element of B can be written as the difference of two positive elements in B then $B^\perp$ is a face of K .

If J is a vector subspace of $A(K)$, the quotient $A(K)/J$ will be ordered by the wedge $\varphi[A(K)^+]$, where $\varphi : A(K) \rightarrow A(K)/J$ is the canonical projection. It is clear that $\varphi(1_K)$ is an order unit for $A(K)/J$.

PROPOSITION III.1.2. The wedge $\varphi[A(K)^+]$ is a cone if and only if J is an order ideal.

If J is an order ideal, let H be an equivalence class in $\varphi[A(K)^+] - \varphi[A(K)^+]$. Then there are $a_1, a_2 \in A(K)^+$ such that $a_1, -a_2 \in H$, so that $0 \leq a_1 \leq a_1 + a_2 \in J$. Thus $a_1 \in J$ and $H = J$, so the wedge is a cone.

If $\varphi[A(K)^+]$ is a proper cone, let $a \leq c \leq b$ with $a, b \in J$. Now $\varphi(c) \in \varphi(a) + \varphi[A(K)^+] = \varphi[A(K)^+]$ and $\varphi(c) \in \varphi(b) - \varphi[A(K)^+] = -\varphi[A(K)^+]$. Hence $\varphi(c) \in \varphi[A(K)^+] \cap -\varphi[A(K)^+]$, so $c \in J$, and J is an order ideal.

Even if J is an order ideal, $A(K)/J$ need not be Archimedean ordered. In the case that it is we have the following:

PROPOSITION III.1.3. If J is an order ideal in $A(K)$ and $A(K)/J$ is Archimedean ordered then $J = J^{\perp\perp}$.

That $J \subset J^{\perp\perp}$ is trivial. Suppose $a \in J$. As $A(K)/J$ is Archimedean, it can be given the order unit norm induced by $\varphi(1_K)$. Let q be a state of $A(K)/J$ such that $q(\varphi(a)) \neq 0$. It follows that $q \circ \varphi$ is a state of $A(K)$ which vanishes on J . I.e. $q \circ \varphi \in J^\perp$, so $a \in J^{\perp\perp}$. Hence $J \supset J^{\perp\perp}$.

If $F \subset K$, $F^{\perp\perp}$ is the set $\{k \in K : a \in A(K) \text{ \& } a|_F = 0 \text{ implies } a(k) = 0\}$. If $F = F^{\perp\perp}$, F is said to be self-determining. The following result is a straightforward application of the Hahn-Banach theorem.

PROPOSITION III.1.4. If $F \subset K$ then $F^{\perp\perp} = L \cap K$, where $L = (F^\perp)^\circ$ is equal to the closed linear span of F in E .

Not all faces of compact convex sets are self-determining:

THEOREM III.1.5. There is a compact convex set with a non self determining closed face.

Let $E = L_2([0,1])$ with the usual order. Define

$$F = \{x \in E : 0 \leq x \leq 1\}$$

$$G = \{x \in E : x \geq 0, \|x\| \leq 1\},$$

and let $a \in E$, $a \geq 0$ be not essentially bounded on $[0,1]$. F and G are weakly compact and convex, and so is $K = \text{co}(F \cup (a + G))$.

As all the members of $\text{aff}(F) = \{\mu x + (1 - \mu)y : x, y \in F, \mu \in \mathbb{R}\}$ are essentially bounded and those of $(a + G)$ are not, $\text{aff}(F) \cap (a + G) = \emptyset$ and F is a face of K .

If f is a bounded affine function on K such that $f|_F = 0$, then $f = 0$, this certainly showing that F is not self-determining. As $\text{aff}(G) = E$, $\text{aff}(K) = E$ and f has a unique affine extension to the whole of E , which we also denote by f . As f vanishes at $0 \in F$, f is linear. But f is bounded on $(a + G)$, and hence on $G - G$, which is a norm neighbourhood of 0 in E . Thus $f \in E^*$. But $\text{lin}(F) = L_\infty$ is norm-dense in E , so $f = 0$.

Proposition III.1.4 shows that if F is a closed face of K , and $\text{lin}(F)$ is closed in E , then F is self-determining. This prompts us to look at closed convex subsets of K which have this property.

Suppose F is a closed convex subset of K , give $\text{lin}(F)$ the relative order induced by E , so that F is a base for the positive cone in $\text{lin}(F)$. Let $\| \cdot \|_F$ denote the base norm in $\text{lin}(F)$ induced by the base F , and $\| \cdot \|$ the subspace norm induced from E .

THEOREM III.1.6. If F is a closed convex subset of K then the following are equivalent:

(1) For the subspace norm $\text{lin}(F)$ is boundedly positively generated. I.e. there is $\lambda \geq 1$ such that for all $q \in \text{lin}(F)$ there are $q_1, q_2 \in \text{lin}(F)$ with $q_1, q_2 \geq 0$ and $\|q_1\| + \|q_2\| \leq \lambda \|q\|$.

(2) $\text{lin}(F)$ is weak* closed in E .

(3) $\text{lin}(F)$ is norm closed in E .

(1) implies that the norms $\| \cdot \|$ and $\| \cdot \|_F$ are equivalent, for if $\text{lin}(F)$ is λ -generated then $\| \cdot \| \leq \| \cdot \|_F \leq \lambda \| \cdot \|$. Now the set $\text{lin}(F) \cap \text{co}(K \cup -K) = (\lambda \text{co}(F \cup -F)) \cap \text{co}(K \cup -K)$ is weak* compact, and the result (2) follows from the Krein-Smulian theorem.

That (2) $\implies$ (3) is trivial.

If (3) holds, note that $\text{lin}(F)$ is complete for the base norm, and that $\| \cdot \| \leq \| \cdot \|_F$. By the open mapping theorem we now have $\| \cdot \|_F \leq \lambda \| \cdot \|$ for some λ , so that (1) is now easily seen to hold.

Let F be a closed convex subset of K . It is easily seen that $A(K)|_F$ may be algebraically identified with $A(K)/F^\perp$. The quotient norm on $A(K)/F^\perp$ corresponds to the norm

$$\|a\|_q = \inf \{ \|\tilde{a}\| : \tilde{a} \in A(K), \tilde{a}|_F = a \}$$

on $A(K)|_F$. The characteristic number of F is defined to be

$$\rho_F = \sup \{ \|a\|_q / \|a\|_F : a \in A(K)|_F, a \neq 0 \},$$

where $\|a\|_F = \sup \{ |a(k)| : k \in F \}$.

The set F is said to have the extension property if $A(F) = A(K)|_F$, i.e. if every $a \in A(F)$ has an extension $\tilde{a} \in A(K)$.

LEMMA III.1.7. Let F be a closed convex subset of K , let $\text{lin}(F)$ be given the base-norm induced by F , and $(F^\perp)^0$ be given the norm induced from E . Then we have the following isometric isomorphisms:

$$A(F)^* \cong \text{lin}(F), \quad (A(K)|_F)^* \cong (F^\perp)^0$$

and

$$\rho_F = \sup \{ \|x\|_F / \|x\| : x \in \text{lin}(F), x \neq 0 \}.$$

The first result we know already. Noting that $A(K)|_F \cong A(K)/F^\perp$, we have

$$\begin{aligned} (A(K)/F^\perp)^* &\cong \{ q \in A(K)^* : q(a) = 0 \text{ for all } a \in F^\perp \} \\ &\cong \{ x \in E : q_x(a) = a(x) = 0 \text{ for all } a \in F^\perp \} \\ &= (F^\perp)^0, \end{aligned}$$

where $x \mapsto q_x$ is the natural injection of E into $A(K)^*$.

The last result follows from these isometries and the Hahn-Banach theorem.

THEOREM III.1.8. If F is a closed convex subset of K , then the following are equivalent:

- (1) F has the extension property.
- (2) $\rho_F < \infty$.
- (3) $\text{lin}(F)$ is boundedly positively generated in the

order and norm induced from E .

(1) $\implies$ (2). $A(F)$ and $A(K)|_F = A(K)/F^\perp$ are complete for their respective norms. If $A(F) = A(K)|_F$ the fact that $\|a\|_F \leq \|a\|_q$ together with the open mapping theorem shows that there exists

λ such that $\|a\|_q \leq \lambda \|a\|_F$ for all $a \in A(F)$. I.e. $\rho_F \leq \lambda < \infty$.

(2) $\Rightarrow$ (3). (2) implies that the subspace norm on $\text{lin}(F)$ is equivalent to its base-norm by III.1.7. As $\text{lin}(F)$ is complete for the base-norm, $\text{lin}(F)$ is closed in E . Now (3) follows from III.1.6.

(3) $\Rightarrow$ (2). (3) implies that $\text{lin}(F)$ is norm closed by III.1.6, so the E -norm and the base-norm on $\text{lin}(F)$ are equivalent, and $\rho_F < \infty$ by III.1.7.

(2) $\Rightarrow$ (1). (2) implies that the quotient norm on $A(K)|_F$ is equivalent to the uniform norm, so that $A(K)|_F$ is uniformly closed in $A(F)$. As $A(K)|_F$ is dense in $A(F)$, $A(K)|_F = A(F)$.

COROLLARY III.1.9. If F is a closed face of K and $\text{lin}(F)$ is boundedly positively generated then F is a self-determining face with the extension property.

Note in particular that if every function in $A(F)$ extends to an element of $A(K)$, then we can find a bound λ such that if $a \in A(F)$ there exists $\tilde{a} \in A(K)$ with $\tilde{a}|_F = a$ and $\|\tilde{a}\| \leq \lambda \|a\|$. In fact we can take any $\lambda > \rho_F$.

Even if we can identify $A(K)|_F$ and $A(F)$ as topological vector spaces, we may not be able to identify $A(K)/F^\perp$ and $A(F)$ as ordered spaces. We show next that if $F = J^\perp$ the relevant condition is the Archimedicity of $A(K)/J$.

PROPOSITION III.1.10. Let J be any order ideal in $A(K)$, and let $F = J^\perp$. Now $A(K)/J$ is Archimedean for the quotient ordering if and only if for each $a_0 \in [A(K)|_F]^+$ there exists $c \in A(K)^+$ with $c|_F = a_0$.

Endow $A(K)/J$ with the order unit norm induced by $\varphi(1_K)$.

If p is a state on $A(K)/J$ then $p \circ \varphi$ is a state on $A(K)$. I.e. there exists $k \in K$ such that $a(k) = (p \circ \varphi)(a)$ for all $a \in A(K)$. Since $a(k) = p(\varphi(a)) = 0$ if $a \in J$, $k \in J^\perp = F$.

Let $a_0 \in [A(K)|_F]^+$, so that $a_0 = b|_F$ for some $b \in A(K)$.

If p is a state of $A(K)/J$ and k the corresponding point of F , then

$$p(\varphi(b)) = b(k) = a_0(k) \geq 0.$$

Hence $\varphi(b) \geq 0$, so by the definition of the quotient ordering there exists $c \in A(K)^+$ with $c \equiv b \pmod{J}$. I.e. $c|_F = b|_F = a_0$, and one implication is proved.

Suppose now that the extension property is satisfied, and that

$$\varphi(a) \leq n^{-1} \varphi(1_K) \quad (n = 1, 2, \dots)$$

we show that $\varphi(a) \leq 0$. There exist $b_n \in J$ such that

$$a \leq n^{-1} 1_K + b_n \quad (n = 1, 2, \dots).$$

Hence

$$a(k) \leq n^{-1} \quad (n = 1, 2, \dots, k \in F)$$

and so $a|_F \leq 0$.

Now there exists $b \in A(K)^+$ such that $b|_F = (-a)|_F$, so that $\varphi(a) = -\varphi(b) \leq 0$, and the result is proved.

An order ideal $J \subset A(K)$ will be called Archimedean if

- (1) $A(K)/J$ is Archimedean ordered.
- (2) J is positively generated.

If also,

- (3) J^0 is positively generated,

then J is termed strongly Archimedean.

PROPOSITION III.1.11. An order ideal $J \subset A(K)$ is Archimedean if and only if it is the kernel of an order homomorphism of $A(K)$ into some Archimedean order unit space.

"Only if" is trivial as $\varphi(A(K))$ is an Archimedean order unit space. Conversely if $\psi: A(K) \rightarrow A'$ is an order homomorphism with kernel J , then J is a positively generated order ideal and $A(K)/J$ is order isomorphic to $\psi(A) \subset A'$. It follows that $A(K)/J$ is Archimedean.

We look now at the faces which correspond to these two properties.

A closed face F of K is Archimedean if $a \in A(F)^+$ and a has any extension $b \in A(K)$, then a has an extension, $\tilde{a}$, in $A(K)^+$ with $\tilde{a} \geq b$. If furthermore every $a \in A(F)^+$ has such an extension, then F is strongly Archimedean.

THEOREM III.1.12. The map $J \mapsto F = J^\perp$ is a bijection of the Archimedean order ideals of $A(K)$ onto the Archimedean faces of K , the inverse being $F \mapsto F^\perp$.

Assume J is Archimedean, so that J is positively generated. III.1.1 now tells us that F is a closed face of K . Let $a \in A(K)$ with $a|_F \geq 0$. III.1.11 implies that there is $b_1 \in A(K)^+$ such that $b_1|_F = a|_F$. Hence $a - b_1 \in J$, and as J is positively generated there is $b_2 \in J^+$ with $b_2 \geq a - b_1$. If $c = b_1 + b_2$, $c \in A(K)^+ + J^+ \subset A(K)^+$ and $a \leq c$. Since $b_2 \in J$ and $a - b_1 \in J$, $c|_F = b_1|_F = a|_F$ so F is Archimedean.

Suppose now that F is a closed Archimedean face of K , then certainly $J = F^\perp$ is a closed order ideal in $A(K)$. Certainly F satisfies the assumptions of III.1.10 so $A(K)/J$ is Archimedean. It remains to prove that J is positively generated. Let $a \in J$, then there exists $c \in A(K)^+$ with $c \geq a$ and $c|_F = a|_F = 0$. Now $a = c - (c - a) \in J^+ - J^+$.

Finally we must show that $F = J^\perp$. Clearly $F \subset J^\perp$. Suppose

$k \in K \setminus F$, then as F is closed the Hahn-Banach theorem yields $a \in A(K)$ such that

$$a(k) < 0 < a(x) \quad \text{for all } x \in F.$$

Now there is $c \in A(K)^+$ with $c|_F = a|_F$, so $a - c \in J$. But $(a - c)(k) < 0$ so $k \notin J^\perp$. The proof is complete.

THEOREM III.1.13. The map $J \mapsto F = J^\perp$ is a bijection of the strongly Archimedean order ideals of $A(K)$ onto the strongly Archimedean faces of K , the inverse being $F \mapsto F^\perp$.

If J is strongly Archimedean, $J^0 = (J^0)^+ - (J^0)^+ = \text{lin}(F)$ is closed in E . Now III.1.6 and III.1.8 imply that F has the extension property, so is strongly Archimedean.

If F has the extension property, III.1.8 shows that $A(F)$ and $A(K)|_F$ are isomorphic. Passing to the dual and using III.1.7 we see that $\text{lin}(F) = (F^\perp)^0 = J^0$. Hence J^0 is positively generated and the proof is complete.

Archimedean faces share the exposedness properties of closed faces of simplexes:

PROPOSITION III.1.14. An Archimedean face $F \subset K$ is relatively exposed.

If $k \in K \setminus F$, by Hahn-Banach separation there is $a \in A(K)$ such that $a(k) < 0$, $a|_F \geq 0$. Now there is $c \in A(K)^+$ such that $c \geq a$ and $c|_F = a|_F$. Now $c - a \in A(K)^+$, $(c - a)|_F = 0$ and $(c - a)(k) > 0$.

COROLLARY III.1.15. An Archimedean face of a metrisable compact convex set is exposed.

Proved as in II.3.3.

A routine application of Edwards' theorem, II.1.5, proves the following:

PROPOSITION III.1.16. Every closed face of a simplex is strongly Archimedean with characteristic number 1.

2 Split faces.

This section looks in detail at the important class of split faces introduced on page 41. Recall that χ_F is the characteristic function of F .

PROPOSITION III.2.1. If F is a closed face of K then for all $k \in K$ there are $\lambda = \hat{\chi}_F(k)$, $x \in F$ and $y \in F'$ such that

$$k = \lambda x + (1 - \lambda)y.$$

Hence $\hat{\chi}_F^{-1}(1) = F$, $\hat{\chi}_F^{-1}(0) = F'$ and $K = \text{co}(F \cup F')$. In particular F' is a G_S -set.

Proposition I.2.6 allows us to find $x \in F$, $y \in K$ with $k = \lambda x + (1 - \lambda)y$ and $\lambda = \hat{\chi}_F(k)$. If $\lambda = 1$ we can choose any $y \in F'$ to achieve the desired decomposition, so suppose $\lambda < 1$. As $\hat{\chi}_F$ is concave:

$$\lambda = \hat{\chi}_F(k) \geq \lambda \hat{\chi}_F(x) + (1 - \lambda) \hat{\chi}_F(y) = \lambda + (1 - \lambda) \hat{\chi}_F(y),$$

so $\hat{\chi}_F(y) = 0$. Suppose $y \notin F'$, so that there is $u \in F$, $v \in K$ and $0 < \mu < 1$ with $y = \mu u + (1 - \mu)v$. Applying $\hat{\chi}_F$ again gives

$$0 \geq \mu + (1 - \mu) \hat{\chi}_F(v) \geq \mu > 0,$$

a contradiction, so $y \in F'$.

The rest of the implications are now clear.

We shall need to know later how different split faces interact. First we prove a simple lemma. Recall that an ordered vector space H is an order direct sum of subspaces I and J if $H = I \oplus J$ and for $i \in I$, $j \in J$, $i + j \geq 0$ if and only if $i \geq 0$ and $j \geq 0$.

LEMMA III.2.2. Let F , F' be convex subsets of K . Then F and F' are a complementary pair of split faces of K if and only if E is an order direct sum of $\text{lin}(F)$ and $\text{lin}(F')$.

If E is such an order direct sum, then it is easily

verified that F and F' are split and complementary. We prove the converse. Suppose $k \in E$, then $k = \alpha k_1 - \beta k_2$ with $\alpha, \beta \geq 0$ and $k_1, k_2 \in K$. Decomposing each k_i between F and F' shows that $k \in \text{lin}(F) + \text{lin}(F')$. We must now show that this decomposition is unique. Suppose $x_i, x'_i \in F, y_i, y'_i \in F' (i = 1, 2)$ and $0 \leq \alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta'$. Suppose also that

$$\alpha x_1 - \beta x_2 + \gamma y_1 - \delta y_2 = \alpha' x'_1 - \beta' x'_2 + \gamma' y'_1 - \delta' y'_2.$$

Then

$$\alpha x_1 + \beta' x'_2 + \gamma y_1 + \delta' y'_2 = \alpha' x'_1 + \beta x_2 + \gamma' y'_1 + \delta y_2.$$

I.e.

$$\begin{aligned} (\alpha + \beta' + \gamma + \delta') \left[\frac{\alpha x_1 + \beta' x'_2 + \gamma y_1 + \delta' y'_2}{\alpha + \beta' + \gamma + \delta'} \right] = \\ (\alpha' + \beta + \gamma' + \delta) \left[\frac{\alpha' x'_1 + \beta x_2 + \gamma' y'_1 + \delta y_2}{\alpha' + \beta + \gamma' + \delta} \right]. \end{aligned}$$

As the terms in square brackets belong to K , which is a base for the positive cone in E , $\alpha + \beta' + \gamma + \delta' = \alpha' + \beta + \gamma' + \delta$. Now the fact that F and F' are split, together with the identity

$$\begin{aligned} \frac{\left(\frac{\alpha x_1 + \beta' x'_2}{\alpha + \beta'} \right) (\alpha + \beta') + \left(\frac{\gamma y_1 + \delta' y'_2}{\gamma + \delta'} \right) (\gamma + \delta')}{\alpha + \beta' + \gamma + \delta'} = \\ \alpha x_1 + \beta' x'_2 + \gamma y_1 + \delta' y'_2, \end{aligned}$$

and the similar one, imply that

$$\alpha + \beta' = \alpha' + \beta$$

$$\gamma + \delta' = \gamma' + \delta$$

and hence

$$\alpha x_1 + \beta' x'_2 = \alpha' x'_1 + \beta x_2$$

$$\gamma y_1 + \delta' y'_2 = \gamma' y'_1 + \delta y_2.$$

I.e. $\alpha x_1 - \beta x_2 = \alpha' x'_1 - \beta' x'_2$ and $\gamma y_1 - \delta y_2 = \gamma' y'_1 - \delta' y'_2$.

Thus the sum is direct. It is clear that a positive element of

E has a positive decomposition, so has only a positive

decomposition. The result is now clear.

PROPOSITION III.2.3. If F and G are proper split faces of K , then every $k \in K$ can be decomposed as

$$k = \alpha_{11}k_{11} + \alpha_{12}k_{12} + \alpha_{21}k_{21} + \alpha_{22}k_{22}$$

where $\alpha_{ij} \geq 0$, $\sum \alpha_{ij} = 1$, $k_{11} \in F \cap G$, $k_{12} \in F \cap G'$, $k_{21} \in F' \cap G$ and $k_{22} \in F' \cap G'$. The α_{ij} 's are uniquely determined, as also is $k_{i_o j_o}$ provided $\alpha_{i_o j_o} \neq 0$.

The existence is obtained by decomposing k between F and F' and then decomposing the components between G and G' . We establish the uniqueness. By the last lemma, $\alpha_{11}k_{11} + \alpha_{12}k_{12}$ is uniquely determined. But then so are $\alpha_{11}k_{11}$ and $\alpha_{12}k_{12}$, and similarly $\alpha_{21}k_{21}$ and $\alpha_{22}k_{22}$. Now these determine α_{ij} and k_{ij} (unless $\alpha_{ij} = 0$).

PROPOSITION III.2.4. If F and G are split faces of K then so are $F \cap G$ and $\text{co}(F \cup G)$. Further $(F \cap G)' = \text{co}(F' \cup G')$ and $\text{co}(F \cup G)' = F' \cap G'$.

If F or G is improper the result is trivial. By the last result it suffices to prove that $(F \cap G)' = \text{co}(F' \cup G')$, to show that $F \cap G$ is split. The second result follows from consideration of the complementary faces.

Suppose $k \in \text{co}(F' \cup G')$ and that $k = \lambda x + (1 - \lambda)y$ with $x \in F \cap G$, $\lambda \neq 0$. Decomposing y by III.2.3 we obtain

$$k = (\lambda x + (1 - \lambda)\alpha_{11}y_{11}) + (1 - \lambda)(\alpha_{12}y_{12} + \alpha_{21}y_{21} + \alpha_{22}y_{22}).$$

Putting $\delta = \lambda + (1 - \lambda)\alpha_{11}$, we have $\delta \neq 0$ since $\lambda \neq 0$. Now

$$u = \delta^{-1}(\lambda x + (1 - \lambda)\alpha_{11}y_{11}) \in F \cap G,$$

and the decomposition

$$k = \delta u + (1 - \lambda)(\alpha_{12}y_{12} + \alpha_{21}y_{21} + \alpha_{22}y_{22})$$

is of the form of III.2.3. As $k \in \text{co}(F' \cup G')$, we must have $\delta = 0$, so that $k \in (F \cap G)'$.

We have a useful characterisation of split faces.

THEOREM III.2.5. If F is a closed face of K , then the following are equivalent:

- (1) F is a split face.
- (2) μ_F is an affine dependence whenever μ is an affine dependence supported by $F \cup F'$.
- (3) μ_F is an affine dependence whenever μ is a boundary affine dependence.

(1) $\implies$ (2). Suppose $\mu = \mu_1 - \mu_2$ where μ_i are probability measures. Let $\alpha_i = \mu_i(F)$, and suppose first that $0 < \alpha_i < 1$ for $i = 1, 2$. Let x_i be the resultant of $\alpha_i^{-1}(\mu_{iF})$ and y_i that of $(1 - \alpha_i)^{-1}(\mu_{iF'})$, so that $x_i \in F$ and $y_i \in F'$. As μ was an affine dependence we have

$$\alpha_1 a(x_1) + (1 - \alpha_1)a(y_1) - \alpha_2 a(x_2) - (1 - \alpha_2)a(y_2) = 0$$

for all $a \in A(K)$. I.e.

$$\alpha_1 x_1 + (1 - \alpha_1)y_1 = \alpha_2 x_2 + (1 - \alpha_2)y_2.$$

But F is split, so $\alpha_1 = \alpha_2$, $x_1 = x_2$ and $y_1 = y_2$. I.e. if $a \in A(K)$

$$(\mu_F)(a) = \alpha_1 a(x_1) - \alpha_2 a(x_2) = 0,$$

and μ_F is an affine dependence.

If e.g. $\alpha_1 = 0$, then $\alpha_2 < 1$ else the common barycentre of μ_1 and μ_2 will lie in $F \cap F' = \emptyset$. Let x_1 be the resultant of μ_1 , and suppose $0 < \alpha_2 < 1$. Defining x_2 and y_2 as before we obtain

$$y_1 = \alpha_2 x_2 + (1 - \alpha_2)y_2,$$

so that $\alpha_2 = 0$ and $y_1 = y_2$. It follows that $\mu_F = 0$. Similar proofs suffice in the other possible degenerate cases.

(2) $\Rightarrow$ (3). Let μ be a boundary affine dependence, then $|\mu|(\hat{\chi}_F) = |\mu|(\chi_F)$. It follows that

$$|\mu|(K \setminus (F \cup F')) = |\mu|(\{k \in K : \hat{\chi}_F(k) > \chi_F(k)\}) = 0,$$

so (2) gives the result.

(3) $\Rightarrow$ (1). Let $k_1, k_2 \in F'$ and μ_i be a boundary probability measure representing k_i . Suppose $0 < \lambda < 1$ and $\lambda k_1 + (1 - \lambda)k_2 = \alpha m + (1 - \alpha)n$ with $0 \leq \alpha \leq 1$, $m \in F$, $n \in F'$. To show that F' is convex we must prove $\alpha = 0$. Let Θ_m, Θ_n be boundary probability measures representing m, n respectively, so that Θ_m is supported by F and Θ_n by F' . Now $\nu = \lambda\mu_1 + (1 - \lambda)\mu_2 - \alpha\Theta_m - (1 - \alpha)\Theta_n$ is a boundary affine dependence, so $\nu_F = -\alpha\Theta_m$ is an affine dependence. This must imply $\alpha = 0$, so F' is convex.

Now suppose

$$\lambda_1 x_1 + (1 - \lambda_1)y_1 = \lambda_2 x_2 + (1 - \lambda_2)y_2$$

where $0 \leq \lambda_i \leq 1$, $x_i \in F$ and $y_i \in F'$ for $i = 1, 2$. Without loss of generality we may assume $\lambda_1 > 0$. For $i = 1, 2$ let μ_i, ν_i be boundary measures representing x_i, y_i , so that $\nu_i(F) = 0$. The measure

$$\pi = (\lambda_1\mu_1 - \lambda_2\mu_2) + ((1 - \lambda_1)\nu_1 - (1 - \lambda_2)\nu_2)$$

is an affine dependence, thus so also is

$$\pi_F = (\lambda_1\mu_1 - \lambda_2\mu_2).$$

I.e. $\lambda_2 = \lambda_1 > 0$, and $\mu_1 - \mu_2$ is an affine dependence, so μ_1 and μ_2 have the same resultant. Hence $x_1 = x_2$ and $y_1 = y_2$.

It follows from the proof of III.2.1 that if F is a split face then $\hat{\chi}_F$ is affine. In fact if $k = \lambda x + (1 - \lambda)y$ with $x \in F$, $y \in F'$ then $\hat{\chi}_F(k) = \lambda$. This does not however characterise split faces. We digress slightly to characterise faces with this property, which we shall need later.

A closed face F of K is termed parallel if F' is a face, and the relation

$$\lambda_1 x_1 + (1 - \lambda_1) y_1 = \lambda_2 x_2 + (1 - \lambda_2) y_2 \quad (x_i \in F, y_i \in F', 0 \leq \lambda_i \leq 1)$$

implies that $\lambda_1 = \lambda_2$.

PROPOSITION III.2.6. A closed face F of K is parallel if and only if $\hat{\chi}_F$ is affine.

If $\hat{\chi}_F$ is affine and $k = \lambda x + (1 - \lambda)y$ with $x \in F, y \in F'$ then $\hat{\chi}_F(k) = \lambda \hat{\chi}_F(x) + (1 - \lambda) \hat{\chi}_F(y) = \lambda$, so the face F is parallel.

Now suppose F is parallel and that $k = \alpha k_1 + (1 - \alpha)k_2$.

We may write

$$k = \lambda x + (1 - \lambda)y$$

$$k_1 = \mu x_1 + (1 - \mu)y_1$$

$$k_2 = \nu x_2 + (1 - \nu)y_2$$

where $x, x_1, x_2 \in F, y, y_1, y_2 \in F'$ and $\lambda = \hat{\chi}_F(k), \mu = \hat{\chi}_F(k_1), \nu = \hat{\chi}_F(k_2)$ by III.2.1. Now we have

$$\begin{aligned} k &= \left[\frac{\alpha \mu x_1 + (1 - \alpha) \nu x_2}{\alpha \mu + (1 - \alpha) \nu} \right] (\alpha \mu + (1 - \alpha) \nu) + \\ &\quad \left[\frac{\alpha (1 - \mu) y_1 + (1 - \alpha) (1 - \nu) y_2}{\alpha (1 - \mu) + (1 - \alpha) (1 - \nu)} \right] (\alpha (1 - \mu) + (1 - \alpha) (1 - \nu)) \\ &= [\alpha \mu + (1 - \alpha) \nu] a + [\alpha (1 - \mu) + (1 - \alpha) (1 - \nu)] b, \end{aligned}$$

where $a \in F$ and $b \in F'$. Hence $\lambda = \alpha \mu + (1 - \alpha) \nu$, i.e.

$\hat{\chi}_F(k) = \alpha \hat{\chi}_F(k_1) + (1 - \alpha) \hat{\chi}_F(k_2)$, so $\hat{\chi}_F$ is affine.

We return now to split faces.

LEMMA III.2.7. $\text{lin}(F) \cap E_1 = \text{co}(F \cup -F)$ if F is a split face.

That $\text{lin}(F) \cap E_1 \supset \text{co}(F \cup -F)$ is obvious. If $k \in \text{lin}(F) \cap E_1$, then $k = \alpha k_1 - (1 - \alpha)k_2$ with $k_1, k_2 \in K$ and $0 \leq \alpha \leq 1$ since $\|k\| \leq 1$. Writing

$$k = \alpha(\lambda x_1 + (1 - \lambda)y_1) - (1 - \alpha)(\mu x_2 + (1 - \mu)y_2)$$

where $x_i \in F$, $y_i \in F'$ and $0 \leq \lambda, \mu \leq 1$, and noting that $k \in \text{lin}(F)$, we see that

$$\alpha(1 - \lambda)y_1 - (1 - \alpha)(1 - \mu)y_2 = 0.$$

We must have $\alpha(1 - \lambda) = (1 - \alpha)(1 - \mu)$ by the uniqueness property of bases. Hence $\alpha\lambda + (1 - \alpha)\mu = 1 - 2\alpha(1 - \lambda) \leq 1$.

It follows that $k = \alpha\lambda x_1 - (1 - \alpha)\mu x_2 \in \text{co}(F \cup -F)$.

COROLLARY III.2.8. F has the extension property if F is a closed split face.

$\text{lin}(F) \cap E_1$ is weak* compact, so $\text{lin}(F)$ is weak* closed by the Krein-Smulian theorem. Now apply III.1.6 and III.1.8.

As $\text{lin}(F)$ is closed, $\text{lin}(F) = (F^\perp)^\circ$. Let P (respectively P') denote the projection of E onto $\text{lin}(F)$ (respectively $\text{lin}(F')$).

PROPOSITION III.2.9. $\text{lin}(F')$ is isometrically isomorphic to $(F^\perp)^*$.

$(F^\perp)^*$ is isometrically isomorphic to $A(K)^*/(F^\perp)^\circ = A(K)^*/\text{lin}(F)$. Also $A(K)^*/\text{lin}(F)$ is isomorphic to $\text{lin}(F')$. That this map is an isometry is because $\|P'\| \leq 1$. This is because

$$\begin{aligned} P'(E_1) &\supset \text{co}(F' \cup -F') \\ &\supset \text{co}(P'(K \cup -K)) \\ &= P'(\text{co}(K \cup -K)) \\ &= P'(E_1). \end{aligned}$$

The proof is complete.

Because of this we may endow $\text{lin}(F')$ with the topology $\sigma(\text{lin}(F'), F^\perp)$, and then apply the Krein-Smulian theorem to this space. This will be our main tool to prove the basic extension theorem below. We must first prove a lemma.

LEMMA III.2.10. Let $\text{lin}(F')$ have the $\sigma(\text{lin}(F'), F^\perp)$ topology, and let Q, R be closed subcones of $\widetilde{F'} \times \mathbb{R}$ in $\text{lin}(F') \times \mathbb{R}$ which do not meet the positive $\mathbb{R}$ -axis. Then $Q + R$ is closed.

Under the norm $\|(a, s)\| = \|a\| + |s|$, $(F^\perp) \times \mathbb{R}$ is a Banach space. By III.2.9 its dual is $\text{lin}(F') \times \mathbb{R}$ with the norm $\|(y, t)\| = \max \{ \|y\|, |t| \}$, the pairing being defined by

$$\langle (a, s), (y, t) \rangle = \langle a, y \rangle + st.$$

The product topology on $\text{lin}(F') \times \mathbb{R}$ is just the $\sigma(\text{lin}(F') \times \mathbb{R}, F^\perp \times \mathbb{R})$ topology. By the Krein-Smulian theorem it will suffice to prove that $(Q + R) \cap E_1$ is compact.

Since $(0, 1)$ is a positive distance from Q and R , there is $k > 0$ such that

$$t \leq k \|y\| \text{ if } (y, t) \in Q \cup R. \quad (*)$$

If $(y, t) \in (Q + R) \cap E_1$, we can write

$$(y, t) = (u, \alpha) + (v, \beta)$$

where $(u, \alpha) \in Q$, $(v, \beta) \in R$ and $\max \{ \|u + v\|, |\alpha + \beta| \} \leq 1$.

As the norm is additive on F' we have $\|u\|, \|v\| \leq 1$, and by using $(*)$ twice we see that

$$\begin{aligned} k \geq k \|y\| &\geq \alpha \geq \alpha + (\beta - k \|v\|) \\ &\geq -|\alpha + \beta| - k \|v\| \\ &\geq -(1 + k), \end{aligned}$$

so that $\|(u, \alpha)\| \leq \max \{ 1, 1 + k \} = 1 + k$. Similarly

$\|(v, \beta)\| \leq 1 + k$. Thus $(y, t) \in S = (1 + k)(Q \cap E_1 + R \cap E_1)$,

which set is compact. Hence $(Q + R) \cap E_1 = S \cap E_1$ is compact and the proof is complete.

THEOREM III.2.11. Let F be a closed split face of K , $a_1, \dots, a_r, b \in A(K)$ and $a_0 \in A(F)$. Suppose $\varepsilon > 0$, $a_i \leq b$ and $a_i|_F \leq a_0 \leq b|_F$. Then there exists $\tilde{a}_0 \in A(K)$ extending a_0 with $a_i \leq \tilde{a}_0 \leq b + \varepsilon$.

By III.2.8 a_0 has at least one extension, so we may suppose $a_0 = 0$ by subtracting this extension from the a_i , b . Now $-a_i, b \geq 0$ on F , we claim that these are l.s.c. on $\tilde{F}'$ for $\sigma(\tilde{F}', F^\perp)$. If $\alpha \in \mathbb{R}$, we must show, for example, that $S = \{y \in \tilde{F}' : \langle b, y \rangle \leq \alpha\}$ is $\sigma(\tilde{F}', F^\perp)$ -closed. It will by the Krein-Smulian theorem suffice to show that

$$\begin{aligned} S_n &= \{y \in S : \|y\| \leq n\} \\ &= \{y \in \tilde{F}' : \langle b, y \rangle \leq \alpha, \langle 1, y \rangle \leq n\} \end{aligned}$$

is compact for each n . But we claim that $S_n = P'(\{k \in K : \langle b, k \rangle \leq \alpha, \langle 1, k \rangle \leq n\})$. The inclusion " $\subset$ " is clear. Suppose $k \in K$, $\langle b, k \rangle \leq \alpha$ and $\langle 1, k \rangle \leq n$, then $\langle b, Pk \rangle \geq 0$ because $Pk \in F$. Thus $\langle b, P'k \rangle = \langle b, k \rangle - \langle b, Pk \rangle \leq \alpha$. Similarly $\langle 1, P'k \rangle \leq n$, so the reverse inclusion is proved. Finally we note that P' is continuous, for if $h \in F^\perp$ then $P'^{-1}(\{y \in \text{lin}(F') : |\langle h, y \rangle| \leq 1\}) = \{x \in A(K)^* : |\langle h, x \rangle| \leq 1\}$ since $\langle h, x \rangle = \langle h, P'x \rangle + \langle h, Px \rangle$ and $h \in F^\perp$. Thus S_n , being the continuous image of a compact set is compact.

We now know that the a_i are u.s.c. and linear, and b is l.s.c. and linear. Let Q_i be the subgraph of a_i on F' , which is hence closed. The last lemma shows that $\sum Q_i$ is closed. Let $G = \{(y, \langle b, y \rangle + \varepsilon) : y \in \text{co}(F' \cup \{0\})\}$, and note that the compact convex set $G^- \subset \{(x, t) : t \geq \langle b, x \rangle + \varepsilon\}$ whilst each Q_i and hence $\sum Q_i \subset \{(x, t) : t \leq \langle b, x \rangle\}$. By Hahn-Banach separation there is a closed hyperplane separating G from $\sum Q_i$, which we may suppose passes through the origin since $\sum Q_i$ is a wedge. This hyperplane is the graph of a function $h \in F^\perp$ such that

$$a_i \leq h \leq b + \varepsilon \quad \text{on } F'.$$

As $h|_F = 0$, $a_i \leq h \leq b + \varepsilon$ on K . The result is now proved.

COROLLARY III.2.12. A closed split face F of K is strongly Archimedean with characteristic number equal to 1.

If $a \in A(K)$, $a|_F \geq 0$ apply III.2.11 with $r = 2$, $a_1 = a$, $a_2 = 0$, $a_0 = a|_F$ and any constant $b > a$. Now $\tilde{a}_0 \in A(K)^+$ and this shows that F is Archimedean.

To see that $\rho_F = 1$, suppose $a_0 \in A(F)$ with $-1 \leq a_0 \leq 1$. Take $r = 1$, $a_1 = -1$, $b = 1$ in III.2.11, to obtain $\tilde{a}_0$ extending a_0 with $\|\tilde{a}_0\| \leq 1 + \xi$ for any $\xi > 0$.

We shall later see that we can in fact obtain norm preserving extensions.

We can also utilise III.2.11 to obtain another characterisation of split faces.

THEOREM III.2.13. A closed face F of a compact convex set K is split if and only if for each $a \in A_S(F)$ [or $A_S(F)^+$, $A(F)$, $A(F)^+$, $A(K)|_F$, $(A(K)|_F)^+$] has an extension $\tilde{a} \in A_S(K)$ such that $\tilde{a}|_{F'} = 0$. If such an extension exists it is unique. In particular if $a \geq 0$ and $a_1|_F = a$, $a_1|_{K \setminus F} = 0$ then $\tilde{a} = \hat{a}_1$.

The uniqueness follows from I.3.2 as $\partial_e K \subset F \cup F'$. Assume F is split and let $a \in A(F)^+$. It will suffice to show that $\hat{a}_1$ is affine, i.e. that the family $\{b \in A(K) : b|_F > a, b > 0\}$ is downward filtering. Suppose $b_1|_F, b_2|_F > a$ and $b_1, b_2 > 0$. We may suppose $(b_1 - 2\xi)|_F, (b_2 - 2\xi)|_F > a$ and $b_1 - 2\xi, b_2 - 2\xi > 0$ for some $\xi > 0$. By Theorem III.2.11 there is an extension $\tilde{a}$ of a with $b_1 - 2\xi, b_2 - 2\xi \geq \tilde{a} \geq -\xi$. I.e. $b_1, b_2 \geq \tilde{a} + 2\xi \geq \xi$. Applying III.2.11 again there is $c \in A(K)$ with $b_1, b_2 \geq c \geq \tilde{a} + \xi \geq 0$. Clearly $c|_F > a$ and $c > 0$, so we have proved the result.

If a is u.s.c. and affine, as well as positive we may apply Propositions I.2.4 and I.2.7 to obtain the result. We

may now obtain the desired extension for differences of such functions. In general we may find a sequence a_n of such differences with $\|a_n - a\| \rightarrow 0$. Using I.3.2 and the fact that $\partial_e K \subset F \cup F'$ we may conclude that

$$\|\tilde{a}_n - \tilde{a}_m\| = \|\tilde{a}_n - \tilde{a}_m\|_{\partial_e K} = \|a_n - a_m\|_{\partial_e K} = \|a_n - a_m\|.$$

So $(\tilde{a}_n)$ is Cauchy in $A_s(K)$. If $\tilde{a} = \lim \tilde{a}_n$ this has the desired property.

Conversely suppose all functions in $(A(K)|_F)^+$ have such extensions. Choosing $a = 1$ we see that $\hat{\chi}_F$ is affine, so F is a parallel face by III.2.6. Suppose $\lambda x_1 + (1 - \lambda)y_1 = \lambda x_2 + (1 - \lambda)y_2$ with $0 < \lambda < 1$, $x_1 \in F$ and $y_1 \in F'$. If $x_1 \neq x_2$ there is $b \in (A(K)|_F)^+$ such that $b(x_1) \neq b(x_2)$. If $\tilde{b}$ is the extension of $b|_F$ that we assume exists we have that $\lambda b(x_1) = \tilde{b}(\lambda x_1) = \tilde{b}(\lambda x_2) + \tilde{b}((1 - \lambda)y_2) - \tilde{b}((1 - \lambda)y_1) = \tilde{b}(\lambda x_2) + 0 + 0 = \lambda b(x_2)$, contradicting the definition of b .

We now prove a second extension theorem, generalising Corollary II.4.3 for simplexes.

PROPOSITION III.2.14. Let F be a closed split face of K , $h \in A(F)$ and $p : K \rightarrow \mathbb{R}$ a strictly positive l.s.c. concave function. If $|h(k)| < p(k)$ for all $k \in F$, then h has an extension $\tilde{h} \in A(K)$ such that $|h(k)| < p(k)$ for all $k \in K$.

We may suppose p is bounded, without loss of generality. Let $R_F : A(K) \rightarrow A(F)$ be the restriction map, and let $G = \{a \in A(K) : |a(k)| < p(k) \text{ for all } k \in K\}$. G is open as p is l.s.c. As R_F maps $A(K)$ onto $A(F)$ by III.2.8, we see that $G|_F$ is open in $A(F)$ by the open mapping theorem. Furthermore $G|_F$ is absolutely convex. Suppose $h \notin G|_F$. By the Hahn-Banach theorem there is a measure ν on K , supported by F , such that

$$\nu(h) > 1 \geq |\nu(a)| \text{ for all } a \in G|_F. \quad (**)$$

Define a norm,

$$\|f\|_p = \sup \{ |f(k)| / p(k) : k \in K \}$$

on $C(K)$, which is equivalent to the supremum norm as p is bounded below and is assumed to be bounded above. The dual norm on $M(K)$ is

$$\|\mu\|_p = \|p\mu\|$$

where $(p\mu)(f) = \mu(pf)$.

By (**) the linear functional $\xi : a \rightarrow \nu(R_F a)$ is $\|\cdot\|_p$ -bounded on $A(K)$, with $\|\xi\|_p \leq 1$. Extending ξ with preservation of $\|\cdot\|_p$ -norm to a bounded linear functional on $C(K)$, we have a measure $\mu \in M(K)$ such that

$$\xi(a) = \mu(a) \text{ (for all } a \in A(K), \|\mu\| = \|\xi\|_p \leq 1. \text{ (***)}$$

Since p is concave $|\tilde{h}(k)| < p(k)$ for all $k \in K$, where $\tilde{h}$ is the extension proved in Theorem III.2.13 to exist in $A_S(K)$. It follows from this and (***) that

$$|\mu(\tilde{h})| = |(p\mu)(p^{-1}\tilde{h})| < 1.$$

(***) and (**) imply that

$$\begin{aligned} \int_K \tilde{h} d\mu &= \int_K \tilde{h} d\nu \\ &= \int_F \tilde{h} d\nu > 1, \end{aligned}$$

contradicting (***). I.e. $h \in G|_F$ and the result is proved.

THEOREM III.2.15. Let F be a closed split face of K , $h \in A(F)$ and $p : K \rightarrow \mathbb{R}$ a strictly positive l.s.c. concave function. If $|h(k)| \leq p(k)$ for all $k \in F$, then h has an extension $\tilde{h} \in A(K)$ such that $|\tilde{h}(k)| \leq p(k)$ for all $k \in K$.

Again we may suppose p is bounded. Applying III.2.14 we may obtain an extension $h_1 \in A(K)$ with $|h_1(k)| < 2p(k)$ for all $k \in K$. Now define $q_2 = 2p \wedge [2^2(p - 2^{-1}|h_1|)]$, which is certainly l.s.c., concave and strictly positive on K , and let h_2 be an extension of h to $A(K)$ with $|h_2| < q_2$.

Assume for induction that we have defined q_m and a_m

($2 \leq m \leq n$) such that

$$|h_m| < 2p \wedge [2^m(p - \sum_{r=1}^{m-1} 2^{-r}|a_r|)] = q_m \quad (m = 2, \dots, n).$$

Let $q_{n+1} = 2p \wedge [2^{n+1}(p - \sum_{r=1}^n 2^{-r}|a_r|)]$, which is l.s.c., concave and strictly positive by the induction hypothesis. Now let

h_{n+1} extend h with $|h_{n+1}| < q_{n+1}$.

We have a sequence $(h_n)_{n=1}^\infty \subset A(K)$ such that

$$(1) \quad h_n|_F = h$$

$$(2) \quad p(k) - \sum_{r=1}^n 2^{-r}|h_r(k)| > 0 \text{ for all } k \in K$$

$$(3) \quad \|h_n\| \leq 2 \sup_{k \in K} p(k).$$

The series $\sum_{r=1}^\infty 2^{-r}h_r$ converges uniformly to a limit $\tilde{h} \in A(K)$.

Clearly $\tilde{h}|_F = h$, and (2) implies that $|\tilde{h}(k)| \leq p(k)$ for all $k \in K$. This completes the proof.

COROLLARY III.2.16. If F is a closed split face of K and $h \in A(F)$ there is $\tilde{h} \in A(K)$ with $\tilde{h}|_F = h$ and $\|\tilde{h}\| = \|h\|$.

We finish this section with an order theoretic characterisation of split faces. As closed split faces are Archimedean, their annihilators are Archimedean ideals in $A(K)$.

PROPOSITION III.2.17. Let J be an Archimedean ideal in $A(K)$ and let $\varphi : A(K) \rightarrow A(K)/J$ be the canonical homomorphism. The face $F = J^\perp$ is split if and only if for every $\varepsilon > 0$ and any $a_1, a_2 \in A(K)^+$:

$$[0, \varphi(a_1)] \wedge [0, \varphi(a_2)] \subset \varphi([0, a_1 + \varepsilon] \wedge [0, a_2 + \varepsilon]),$$

where $[b, c] = \{d : b \leq d \leq c\}$.

This formula means precisely that for every $a_0 \in A(F)^+$ such that $a_0 \leq a_1|_F, a_2|_F$ there is $c \in A(K)^+$ with $c|_F = a_0$, $c \leq a_1 + \varepsilon$ and $c \leq a_2 + \varepsilon$. If F is split, this follows from III.2.11. Conversely if this holds, the family $\{a \in A(K)^+ : a > c\}$

is downward filtering for all $c \in A(F)^+$, i.e. if $b|_F \in A(F)^+$, $b|_{K \setminus F} = 0$ then $\hat{b}$ is affine. By III.2.13 F is split.

Such ideals have been termed near lattice ideals. It has been shown that it is not in general possible to take $\xi = 0$ in this formula.

3 Banach space properties of $A(K)$.

In this section we look at various properties of Banach spaces that are of interest, and relate these to properties of K . The first result that we prove is an analogue of the Kakutani-Stone-Weierstrass theorem for $C(\mathcal{K})$. We describe the closure of certain subspaces of $A(K)$, and derive a sufficient condition for a subspace to be dense. In fact we discuss two different conditions, and discuss their uses later.

L will denote a subspace of $A(K)$ containing the constants.

If $k \in K$ we define

$$T(k) = \{k' \in K : f(k') = f(k) \text{ for all } f \in L\}.$$

L will also be assumed to have the weak R.S.P., i.e. if $f_1, f_2 < h_1, h_2$ then there is g with $f_1, f_2 < g < h_1, h_2$, and to satisfy:

For all $\varepsilon > 0$, $k \in \partial_e K$ and $f \in L$ with $f(k) = 0$ there is $g \in L$ with $g \geq f, 0$ and $g(k) < \varepsilon$.

Such a subspace will be termed an admissible subspace of $A(K)$.

LEMMA III.3.1. If L is an admissible subspace of $A(K)$ and $k \in \partial_e K$ then $T(k)$ is a face of K .

Suppose $k_1 = (1/2)(k_2 + k_3)$, $k_1 \in T(k)$, $k_2 \notin T(k)$. As L contains the constants, there is $f \in L$ with $f(k_1) = 0$, $f(k_2) > 2\varepsilon > 0$. Now as $k_1 \in T(k)$, $f(k) = 0$ there is $g \in L$ with $g(k) = g(k_1) < \varepsilon$, $g(k_2) \geq f(k_2) > 2\varepsilon$, and $g(k_3) \geq 0$. Now $g(k_1) = (1/2)g(k_2) + (1/2)g(k_3) > \varepsilon$, a contradiction.

Such faces will be termed L -faces of K . If $f \in C(K)$ we define

$$\bar{f}(k) = \inf \{g(k) : L \ni g \geq f\}.$$

LEMMA III.3.2. If L is an admissible subspace of $A(K)$, $f \in C(K)$, and $k \in \partial_e K$, then $\bar{f}$ is constant on $T(k)$ with value $\sup \{f(k') : k' \in T(k)\}$.

If $k' \in T(k)$ then $\bar{f}(k') = \inf \{g(k') : L \ni g \geq f\} = \inf \{g(k) : L \ni g \geq f\} = \bar{f}(k)$, so $\bar{f}$ is constant on $T(k)$. Clearly $\bar{f}(k) \geq \sup \{f(k') : k' \in T(k)\}$. Given $\mu \in P(K)$, a proof similar to that of Proposition I.3.4 yields $\nu \in P(K)$ with $\nu \succ \mu$ and $\nu(f) = \mu(\bar{f})$. As $r(\nu) = r(\mu)$, $\nu \in P_k(L) = \{\mu \in P(K) : \mu(l) = l(k) \text{ for all } l \in L\}$ whenever $\mu \in P_k(L)$. Now $\bar{f}(k) = \inf \{g(k) : L \ni g \geq f\} = \max_{\mu \in P_k(L)} \inf \{\mu(g) : L \ni g \geq f\} = \max_{\mu \in P_k(L)} \mu(\bar{f}) = \max_{\mu \in P_k(L)} \mu(f) \leq \sup \{f(k') : k' \in T(k)\}$ because if $\mu \in P_k(L)$ then $\text{supp}(\mu) \subset T(k)$.

THEOREM III.3.3. Let L be an admissible subspace of $A(K)$. Then

$$\bar{L} = \{f \in A(K) : f \text{ is constant on each } L\text{-face}\}.$$

It is clear that $f \in \bar{L}$ implies that f is constant on each L -face. Suppose f is constant on each L -face.

The family $\mathcal{E} = \{g \in L : g < -f\}$ is upward filtering. For let $g_1, g_2 \in \mathcal{E}$ and $u = g_1 \vee g_2$. As L has the weak R.S.P. the family $\mathcal{F} = \{v \in L : v \geq u\}$ is downward filtering, so $\bar{u}$ is u.s.c. and affine. Moreover if $k \in \partial_e K$, $\bar{u}(k) = u(k)$ by III.3.2, so that $\bar{u}(k) \leq -f(k)$ for all $k \in \partial_e K$. But $f + \bar{u}$ is u.s.c. and convex so attains its supremum on $\partial_e K$, so

$$u(k) \leq \bar{u}(k) < -f(k) \text{ for all } k \in K.$$

Thus for all $k \in K$ there is $v_k \in \mathcal{F}$ with $v_k(k) < -f(k)$. Now for all k' in a neighbourhood G_k of k we have

$$u(k') < v_k(k') < -f(k).$$

By compactness there are finitely many $k_i \in K$ such that

$$u < v_{k_1} \wedge v_{k_2} \wedge \dots \wedge v_{k_n} = w < -f.$$

Now as L has the weak R.S.P. there is $g \in L$ with $u = g_1 \vee g_2 < g < w < -f$.

I.e. $\mathcal{E}$ is upward filtering.

Thus $\mathcal{E}$ converges to an u.s.c. affine function $\bar{f}$. We claim $\bar{f} = f$. It being clear that $\bar{f} \geq f$, suppose $g \geq f$. III.3.2 shows that $g(k) \geq \bar{f}(k)$ if $k \in \partial_e K$, so $g(k) \geq \bar{f}(k)$ for all $k \in K$ by the Krein-Milman theorem. Now we may apply Dini's theorem to see that f is in the closure of L .

COROLLARY III.3.4. If L is an admissible subspace of $A(K)$ which separates the points of $\partial_e K$, then L is dense in $A(K)$.

In order to state a slightly different form of this, we define a subspace L of $A(K)$ to be weakly filtering if $f, g \in L$, $a \in A(K)$ and $f, g < a$ imply there is $h \in L$ with $f, g < h < a$. It is easily verified that for such an L , containing the constants, $T(k)$ is again a face of K if $k \in \partial_e K$.

This condition may be used directly to prove the following result, shortcutting most of the last proof.

THEOREM III.3.5. Let L be a weakly filtering subspace of $A(K)$ containing the constants. Then $\bar{L} = \{f \in A(K) : f \text{ is constant on every } L\text{-face}\}$.

COROLLARY III.3.6. Let L be a weakly filtering subspace of $A(K)$, containing the constants and separating the points of $\partial_e K$. Then L is dense in $A(K)$.

The first formulation has the advantage that it only involves properties of L itself, whilst the second uses the relationship between L and $A(K)$. However the latter is rather more general, the weak R.S.P. condition limiting the usefulness of Corollary III.3.4 to the case when K is a simplex.

We turn now to direct sum decompositions of $A(K)$ -spaces.

Recall that E is a direct sum of subspaces E_1, E_2 if each $x \in E$ can be written uniquely as $x_1 + x_2$ with $x_i \in E_i$. We then write $E = E_1 \oplus E_2$. We shall also write $E = E_1 \oplus E_2$ if E is a direct sum of subspaces isomorphic to E_i . If E is a Banach space, we say that E is an M-direct sum of E_1 and E_2 , and write $E = E_1 \oplus_M E_2$, if also $x_i \in E_i$ implies $\|x_1 + x_2\| = \max \{ \|x_1\|, \|x_2\| \}$. It is easily verified that each E_i must be complete.

We first prove a lemma.

LEMMA III.3.7. Let K be a compact convex set, C a closed subset of K such that $K = \text{co}(C)$ and E a locally convex space. If a is a continuous function from C into E which has a unique affine extension $\tilde{a}$ to the whole of K , then $\tilde{a}$ is continuous.

Let $\Phi : C \times C \times [0, 1] \rightarrow K$ map $(y, z, \lambda) \mapsto \lambda y + (1 - \lambda)z$, and $\Psi : C \times C \times [0, 1] \rightarrow E$ map $(y, z, \lambda) \mapsto \lambda a(y) + (1 - \lambda)a(z)$. Clearly both Φ and Ψ are continuous. The assumption that a has a unique affine extension to K , $\tilde{a}$, amounts to saying that $\Psi(\Phi^{-1}\Phi(y, z, \lambda))$ is a singleton for each point of $C \times C \times [0, 1]$.

If D is a closed subset of E , then $\tilde{a}^{-1}(D) = \Phi(\Psi^{-1}(D))$ is closed. This is because Ψ is continuous so that $\Psi^{-1}(D)$ is closed in $C \times C \times [0, 1]$ and is hence compact. Then $\Phi(\Psi^{-1}(D))$ is compact in K and hence closed. Thus $\tilde{a}$ is continuous as claimed.

PROPOSITION III.3.8. Let F_1, F_2 be closed, complementary split faces of K . Then $A(K) = A(F_1) \oplus_M A(F_2)$.

Suppose $a_i \in A(F_i)$, the fact that F_i are complementary implies that $K = \text{co}(F_1 \cup F_2)$, and that F_i are split implies there is an affine function a on K which coincides with a_i on F_i .

Namely if $k \in K$ and $k_i \in F_i$, $0 \leq \lambda \leq 1$ with $k = \lambda k_1 + (1 - \lambda)k_2$ then $a(k) = \lambda a_2(k_1) + (1 - \lambda)a_2(k_2)$ is the required function. Lemma III.3.7 implies that a is continuous. This shows that $A(F_1) \oplus A(F_2)$ is a subspace of $A(K)$, and it is clearly all of $A(K)$. The uniqueness of the sum and the condition on the norm are both direct consequences of the Krein-Milman theorem.

We shall see that this is the only way in which M -direct sum decompositions can arise. We do this via the dual space. We say that E is an L -direct sum of E_1, E_2 , and write $E = E_1 \oplus_L E_2$, if E is a direct sum of E_1 and E_2 and $\|x_1 + x_2\| = \|x_1\| + \|x_2\|$ whenever $x_i \in E_i$.

LEMMA III.3.9. If $A = A_1 \oplus_M A_2$, then $A^* = A_1^{\circ} \oplus_L A_2^{\circ}$.

That $A_1^{\circ} + A_2^{\circ} \subset A^*$ is direct from the definitions. If $f \in A^*$, define f_1 on A by $f_1(x) = f(x)$ if $x \in A_2$
 $= 0$ if $x \in A_1$.

As A_i are closed, f_1 is bounded and lies in A_1° . We may similarly define $f_2 \in A_2^{\circ}$, and find that $f = f_1 + f_2$. I.e. $A^* = A_1^{\circ} + A_2^{\circ}$. The sum is direct because if f vanishes on both A_1 and A_2 it must be identically zero.

Let $f_1 \in A_1^{\circ}$, $f_2 \in A_2^{\circ}$ then

$$\begin{aligned} \|f_1 + f_2\| &= \sup \{ |(f_1 + f_2)(x)| : x \in A, \|x\| \leq 1 \} \\ &= \sup \{ |f_1(x_1) + f_2(x_2)| : x_i \in A_i, \|x_i\| \leq 1 \} \\ &= \sup \{ |f_1(x_1)| + |f_2(x_2)| : x_i \in A_i, \|x_i\| \leq 1 \} \\ &= \|f_1\| + \|f_2\|. \end{aligned}$$

I.e. $A^* = A_1^{\circ} \oplus_L A_2^{\circ}$.

PROPOSITION III.3.10. If E is an ordered vector space equipped with a base norm, and $E = E_1 \oplus_L E_2$, then E_i are ideals in E .

Let $x_1 \in E_1$ and suppose that $0 \leq y_1 + y_2 \leq x_1$, with $y_i \in E_i$.

Let $z_1 + z_2 = x_1 - (y_1 + y_2) \geq 0$, with $z_i \in E_i$. As the norm is additive on the positive cone of E , we have

$$\begin{aligned} \|x_1\| &= \|y_1 + y_2\| + \|z_1 + z_2\| \\ &= \|y_1\| + \|y_2\| + \|z_1\| + \|z_2\|, \end{aligned}$$

since E is an L-direct sum of the E_i . As the sum is direct,

$y_2 + z_2 = 0$, so that $x_1 = y_1 + z_1$. Then we have

$$\begin{aligned} \|y_1\| + \|z_1\| + 2\|y_2\| &= \|x_1\| \\ &= \|y_1 + z_1\| \\ &\leq \|y_1\| + \|z_1\|. \end{aligned}$$

I.e. $y_2 = z_2 = 0$, and thus E_1 is an order ideal. Similarly E_2 is an order ideal.

We must now show that E_i is positively generated. Let $B = \{x \in E : x \geq 0, \|x\| = 1\}$. We claim that $E_1 \cap B$ and $E_2 \cap B$ are complementary split faces of B . Let $x \in B$, then $2x = x_1 + x_2$ with $x_i \in E_i$ and $2 = 2\|x\| = \|x_1\| + \|x_2\|$. If either $x_i = 0$ then x belongs to $E_1 \cap B$ or $E_2 \cap B$. If neither term is zero, we have

$$x = (\|x_1\|/2)(x_1/\|x_1\|) + (\|x_2\|/2)(x_2/\|x_2\|),$$

with $\|x_i/\|x_i\|\| = 1$ and $\|x_1\|/2 + \|x_2\|/2 = 1$. It is now clear that the faces are complementary. That the faces are split follows from the fact that the sum is direct. If $S_i = \text{co}((E_i \cap B) - (E_i \cap B))$, then it is readily seen that S_i generates all of E_i , for if this were not so, $S = \text{co}(S_1 \cup S_2)$ would not generate all of E . But S contains the open unit ball of E so generates the whole space. Hence each E_i is an ideal as required.

THEOREM III.3.11. Suppose $A(K) = A_1 \oplus_M A_2$. Then there are complementary closed split faces F_1 and F_2 of K such that A_1 is isometric to $A(F_1)$.

A_i^0 is closed, hence so are the sets $F_j = A_i^0 \cap K$ ($j \neq i$, $i, j = 1, 2$). By III.3.9 and III.3.10 F_1 and F_2 have linear spans for which $A(K)^*$ is an order direct sum of these two subspaces. It follows from III.2.2 that F_1 and F_2 are complementary split faces of K . Now $A_i = (A_i^0)^\perp = F_j^\perp = \{a \in A(K) : a|_{F_j} = 0\}$ is isometric to $A(F_1)$ by the proof of III.3.8.

We turn now to isometric injections of $A(K_1)$ into $A(K_2)$. We look first at the case when this is an isometry onto. Again we first describe a way in which isometric isomorphisms can arise, and then show that this is the only possible type.

PROPOSITION III.3.12. Let K be a compact convex set, and F and G be complementary closed parallel faces of K . Let $g_0 \in \text{aff}(G)$ and let $G' = \{2g_0 - g : g \in G\}$. Then $K' = \text{co}(F \cup G')$ is a compact convex set, and $A(K')$ is linearly isometric to $A(K)$.

It is clear that K' is a compact convex set. Define $T : A(K) \rightarrow A(K')$ by

$$\begin{aligned} Ta|_{F_1} &= a|_{F_1} \\ Ta(2g_0 - g) &= -a(g) \quad (g \in G) \end{aligned}$$

(as F and G are parallel this has an affine extension to K' which is continuous by III.3.7). Clearly T is onto, and is an isometry because each a attains its maximum modulus on $F \cup G$.

THEOREM III.3.13. Let K, K' be compact convex sets such that $A(K)$ is isometrically isomorphic to $A(K')$. Then there are closed, complementary parallel faces F, G of K such that if $g_0 \in \text{aff}(G)$ and $G' = \{2g_0 - g : g \in G\}$ then K' is affinely homeomorphic to $\text{co}(F \cup G')$.

Let $A = A(K) = A(K')$, then K and K' are affinely homeomorphic to proper closed faces of A_1^* , the closed unit ball of A^* with the weak* topology, such that $\text{co}(K \cup -K) = \text{co}(K' \cup -K') = A_1^*$ (identifying K, K' with their images in A_1^*). It will suffice to prove that two such faces are related in the manner described.

Let $F = K \cap K', G = K \cap (-K')$. If $k \in \partial_e K$ then $k \in \text{co}(K' \cup -K')$ by assumption. As k is extreme in A_1^* , $k \in K' \cup -K'$ so $k \in F \cup G$. Now the Krein-Milman theorem assures us that $K = \text{co}(F \cup G)$. It is similarly seen that $K' = \text{co}(F \cup -G)$. It is clear that F, G are closed faces of K . Suppose

$$\lambda f_1 + (1 - \lambda)g_1 = \mu f_2 + (1 - \mu)g_2$$

$f_1 \in F, g_1 \in G$ and $0 \leq \lambda, \mu \leq 1$. If we operate on this identity by the function that is identically 1 on K' (and hence 1 on F , -1 on G) we see that

$$\lambda - (1 - \lambda) = \mu - (1 - \mu).$$

I.e. $\lambda = \mu$, so the faces are parallel. Now choose $g_0 \in \text{aff}(G)$ then $2g_0 - G$ does not meet F . As F and G are parallel there is an affine extension of the map that is the identity on F and that is $g \mapsto 2g_0 - g$ on G . By III.3.7 this is continuous. As this is clearly (1,1) and K is compact, it is a homeomorphism and the proof is complete.

COROLLARY III.3.14. Let K be a compact convex set and S a simplex. If $A(K)$ is linearly isometric to $A(S)$ then K is affinely homeomorphic to S .

As S is a simplex F and G are split (by II.3.9). Thus there is a (1,1) continuous (by III.3.7) affine map of $K = \text{co}(F \cup -G)$ onto $S = \text{co}(F \cup G)$. I.e. K is affinely homeomorphic to S .

We close by describing isometric injections of $A(K)$ into $A(S)$, for S a simplex and K an arbitrary compact convex set. There are three basic ways in which these can arise:

Let $\pi : S \longrightarrow K$ be a continuous affine surjection, and define $A(\pi) : A(K) \longrightarrow A(S)$ by $(A(\pi)a)(k) = a(\pi k)$ for $a \in A(K)$, $k \in S$. $A(\pi)$ is a positive linear operator taking 1_K to 1_S , which is an isometry as π is onto.

Let $S = \text{co}(F \cup G)$ where F, G are disjoint closed faces of S , and define $e \in A(S)$ to be 1 on F and -1 on G (this exists as F and G are split by II.3.9 and using III.3.7). If $a \in A(S)$ we define $e \circ a$ to be that element of $A(S)$ with $e \circ a(k) = e(k)a(k)$ for all $k \in \partial_e S$ (again by II.3.9 and III.3.7). The map $a \longmapsto e \circ a$ is a linear isometry of $A(S)$ onto itself.

Let F be a closed face of S , and let $H : A(F) \longrightarrow A(S)$ be a norm one linear extension operator, not necessarily positive or preserving the constants. Then H is an isometry.

We show that every isometric injection can be decomposed into (basically) these three.

THEOREM III.3.15. Let K be a compact convex set and S a simplex. Suppose $T : A(K) \longrightarrow A(S)$ is an isometric injection. There is a closed face F of S and a continuous affine surjection $\pi : F \longrightarrow K$; a function $e \in A(F)$ that is 1 on some closed face of F and -1 on its complementary face (which is also closed); and a linear extension operator $H : L \longrightarrow A(S)$ where $L = \{Ta|_F : a \in A(K)\} \subset A(F)$ such that

$$Ta = H(e \circ (A(\pi)(a))).$$

Let F^+ (respectively F^-) = $\{k \in S : (T1_K)(k) = +1 \text{ (resp. } -1)\}$, then F^+ and F^- are the sets where the continuous affine function $T1_K$ attains its supremum and infimum respectively, and hence are faces. By II.3.7 F^+ and F^- are split faces, so

$\text{co}(F^+ \cup F^-) = F$ is a face by III.2.4. We claim that the map $a \mapsto Ta|_F$ is an isometry. If not suppose $\|Ta|_F\| < \|a\|$, and that a attains its maximum modulus at $x_0 \in K$, where $a(x_0) = \|a\|$ (the proof when $a(x_0) = -\|a\|$ is similar). Then we have:

$$\begin{aligned} \|a\| + 1 &= \|a + 1\| = \|T(a + 1)\| \\ &\leq \|Ta\| + \|T1\| \\ &< \|a\| + 1, \end{aligned}$$

for off F , $|T1| < 1$ and on F $|Ta| < 1$ by assumption. As this is impossible $a \mapsto Ta|_F$ is an isometry. Let $e = 1$ on F^+ and $e = -1$ on F^- with $e \in A(F)$. Then it is clear that $a \mapsto e \circ (Ta|_F)$ is still an isometry, and now $1_K \mapsto 1_F$ so this is an order homomorphism from $A(K)$ into $A(F)$. By I.1.20 there is a continuous affine $\pi: F \rightarrow K$ such that this operator is $A(\pi)$. π is onto as $A(\pi)$ is an isometry. It is clear that $Ta|_F = e \circ (A(\pi)(a))$. H is now define by $H(Ta|_F) = Ta$ when $a \in A(K)$, and the proof is complete.

CHAPTER IV

FACIAL TOPOLOGIES

1 Split face topology.

We have seen that the collection of closed split faces of a compact convex set K is closed under finite convex hulls and finite intersections. More is in fact true.

PROPOSITION IV.1.1. The collection of closed split faces of K is closed under finite convex hulls and arbitrary intersections.

That the collection is closed under finite convex hulls has already been proved, III.2.4. This result also means that it suffices to prove that the intersection, F , of a downwards filtering family of closed split faces $\{F_\alpha\}$ is a closed split face. It will be enough to prove that if $a_0 \in A(F)^+$ the set $\{a \in A(K) : a > 0, a|_F > a_0\}$ is downwards filtering, for then the pointwise infimum of the set lies on $A_S(K)$ and extends a_0 , so F is split by III.2.13. Suppose a_1, a_2 belong to this set. Let g be defined on K by $g|_F = a_0$, $g|_{K \setminus F} = 0$. By I.2.2 we have

$$\hat{g}(k) < (a_1 \wedge a_2)^\vee(k) \text{ for all } k \in F.$$

The set $U = \{k \in K : \hat{g}(k) < (a_1 \wedge a_2)^\vee(k)\}$ is open and contains F . By compactness there is an α such that $F_\alpha \subset U$, and by Hahn-Banach separation there is $a \in A(F_\alpha)$ such that

$$\hat{g}|_{F_\alpha} < a < (a_1 \wedge a_2)^\vee|_{F_\alpha}.$$

Hence we must have

$$g|_{F_\alpha} < a < a_1|_F, a_2|_F.$$

As the inequalities are strict we may apply III.2.11 to obtain $b \in A(K)$ with $b|_{F_\alpha} = a$, $0 < b < a_1, a_2$. Since $b|_{F_\alpha} = a > \hat{g}|_{F_\alpha}$ this proves the directedness, and the result. -

It follows that the sets $F \cap \partial_e K$ for F a closed split face of K form the closed sets of a topology on $\partial_e K$, the

$\mathcal{Q}$ -topology. The collection is trivially closed under intersections, and under finite unions because:

$$(F_1 \cap \partial_e K) \cup (F_2 \cap \partial_e K) = \text{co}(F_1 \cup F_2) \cap \partial_e K.$$

PROPOSITION IV.1.2. $\partial_e K$ is compact for the $\mathcal{Q}$ -topology.

Let (S_α) be a collection of $\mathcal{Q}$ -closed sets in $\partial_e K$, with the finite intersection property. If $S_\alpha = F_\alpha \cap \partial_e K$ for F_α a closed split face of K , then the family (F_α) has the finite intersection property. It follows that $F = \bigcap_\alpha F_\alpha \neq \emptyset$. Now $\emptyset \neq \partial_e F \subset \partial_e F_\alpha = S_\alpha$ for all α , so $\bigcap_\alpha S_\alpha \neq \emptyset$, and the proof is complete.

It is certainly not however true that $\partial_e K$ is in general Hausdorff. We shall return to this point later. We first derive some useful extension properties of functions defined on $\partial_e K$.

THEOREM IV.1.3. If $f : \partial_e K \longrightarrow \mathbb{R}$ is bounded and $\mathcal{Q}$ -u.s.c., and $a \in A(K)^+$ then there is a unique u.s.c. affine function $c : K \longrightarrow \mathbb{R}$ such that

$$c(k) = f(k)a(k) \text{ for all } k \in \partial_e K.$$

If $f : \partial_e K \longrightarrow \mathbb{R}$ is $\mathcal{Q}$ -continuous and $a \in A(K)$ there is $c \in A(K)$ such that

$$c(k) = f(k)a(k) \text{ for all } k \in \partial_e K.$$

We assume first that $f(\partial_e K) \subset [0, 1]$, and we shall construct a descending sequence (c_n) of u.s.c. affine functions which converge to $f(a|_{\partial_e K})$ on $\partial_e K$.

For $n = 1, 2, \dots$, $i = 0, 1, 2, \dots, 2^n$ let

$$C_{n,i} = \{k \in \partial_e K : f(k) \geq 2^{-n} i\}.$$

By hypothesis there are closed split faces $F_{n,i}$ of K such that

$$\begin{aligned} C_{n,i} &= \partial_e K \cap F_{n,i}. \text{ Define } b_{n,i}(k) = 2^{-n} a(k) \quad (k \in F_{n,i}) \\ &= 0 \quad (k \in K \setminus F_{n,i}). \end{aligned}$$

Let $a_{n,i} = \hat{b}_{n,i}$, which is affine by III.2.13. As $b_{n,i}$ is u.s.c. we have

$$a_{n,i}(k) = 2^{-n}a(k) \quad (k \in C_{n,i})$$

$$= 0 \quad (k \in \partial_e K \setminus C_{n,i}).$$

Now define $c_n = \sum_{i=1}^{2^n} a_{n,i}$, a sequence of u.s.c. affine functions. Now $c_n|_{\partial_e K} \searrow f(a|_{\partial_e K})$, hence (I.2.2) (c_n) is also a descending sequence. Now $c = \inf c_n$ satisfies the conditions of the theorem.

If f is bounded, choosing a suitable multiple and adding a suitable constant enables us to reduce the result to this case. The uniqueness follows from I.2.2.

Now suppose f is $\mathbb{Q}$ -continuous, and let $-\lambda \leq a \leq \lambda$. By the first part of the proof there are u.s.c. affine functions c_1 and c_2 on K such that for all $k \in \partial_e K$,

$$c_1(k) = f(k)(a(k) + \lambda), \quad c_2(k) = -f(k).$$

The function $c = c_1 + \lambda c_2$ is u.s.c. and affine and satisfies

$$c(k) = f(k)a(k) \quad (k \in \partial_e K).$$

If g is u.s.c. and concave with $g \geq f(a|_{\partial_e K})$ then $g \geq c$ (I.2.2). It follows that $c = (f(a|_{\partial_e K}))^\wedge$. Similarly $d = (f(a|_{\partial_e K}))^\vee$ is affine.

Now $c - d$ is an u.s.c. affine and positive function. As $c|_{\partial_e K} = d|_{\partial_e K}$, $c = d$. It follows that c is continuous as required. Uniqueness is trivial.

Taking $a = 1_K$ we obtain:

COROLLARY IV.1.4. If $f : \partial_e K \rightarrow \mathbb{R}$ is $\mathbb{Q}$ -continuous, then f has a unique extension to an element of $A(K)$.

THEOREM IV.1.5. Let K be a compact convex set, then the following are equivalent:

- (1) K is a Bauer simplex.
- (2) The $\mathbb{Q}$ -topology on $\partial_e K$ is Hausdorff.
- (3) The $\mathbb{Q}$ -topology and relative topology on $\partial_e K$ coincide.

(1) $\implies$ (2). If $G \subset \partial_e K$ is relatively closed, then by II.2.5 $G = \partial_e \overline{\text{co}}(G)$ and $\overline{\text{co}}(G)$ is a closed face of K , and is split by II.3.7. It follows that G is $\mathbb{Q}$ -closed. Thus (2) is certainly true.

(2) $\implies$ (3). First we show $\partial_e K$ is closed in K . If $k \in \overline{\partial_e K} \setminus \partial_e K$, let F_k be the smallest closed split face containing k . As $F_k \neq \{k\}$, by the Krein-Milman theorem there exist distinct extreme points u and v of F_k . Let U and V be $\mathbb{Q}$ -open subsets of $\partial_e K$ with $u \in U$, $v \in V$ and $U \cap V = \emptyset$, which is possible by the Hausdorff assumption. Let G and H be closed split faces of K such that $\partial_e G = \partial_e K \setminus U$ and $\partial_e H = \partial_e K \setminus V$. Now $\partial_e G \cup \partial_e H = \partial_e K$, so $k \in \overline{\partial_e K} = \overline{\partial_e G \cup \partial_e H} \subset G \cup H$. Suppose $k \in G$, then $F_k \subset G$ and $\partial_e F_k \subset \partial_e G$. I.e. $u, v \in \partial_e G = \partial_e K \setminus U$ - a contradiction. Now the identity map from $\partial_e K$ with the relative topology into $\partial_e K$ with the $\mathbb{Q}$ -topology is continuous, (1,1) and onto. As $\partial_e K$ is compact for the relative topology, the topologies coincide.

(3) $\implies$ (1). By IV.1.4 every continuous function on $\partial_e K$ extends to an element of $A(K)$. Thus K is a Bauer simplex by II.2.3.

The set of extensions to $A(K)$ of $\mathbb{Q}$ -continuous functions on $\partial_e K$ is a linear subspace of $A(K)$ that we denote by $\tilde{A}(K)$. We establish a characterisation of this subspace.

The centre of $A(K)$ is the set of all $b \in A(K)$ such that for every $a \in A(K)$ there exists $c \in A(K)$ with

$$c(k) = a(k)b(k) \text{ for all } k \in \partial_e K.$$

From IV.1.3 it follows that this will contain $\widetilde{A(K)}$. We shall see that they are in fact identical. First we deduce a few properties of the centre. We recall that a lattice ordered Banach space is an M-space if $|x| \leq |y| \implies \|x\| \leq \|y\|$ and $\| |x| \vee |y| \| = \max \{ \|x\|, \|y\| \}$ for all x and y . We insert first a well-known lemma.

LEMMA IV.1.6. Let $1 \in C$ be a closed subalgebra of the space of all bounded functions on a set X , $B(X)$. Then C is a sublattice of $B(X)$.

It will suffice to prove that $f \in C$ implies $|f| \in C$. Let $I = \{f(x)^2 : x \in X\}$, and let P_n be a sequence of polynomials such that $P_n(z) \longrightarrow \sqrt{z}$ uniformly on the compact interval I . Let $g_n(x) = P_n(f(x)^2)$ ($x \in X$). Then it is easily seen that (g_n) is a Cauchy sequence converging to $g \in C$ satisfying

$$g(x) = \sqrt{f(x)^2} = |f(x)| \quad (x \in X).$$

I.e. $g = |f| \in C$.

PROPOSITION IV.1.7. The centre of $A(K)$ is a commutative Banach algebra and an M-space for the order and norm induced from $A(K)$, and the multiplication $(a, b) \longmapsto a \circ b \in A(K)$ where

$$(a \circ b)(k) = a(k)b(k) \quad (k \in \partial_e K).$$

Further the lattice operations satisfy

$$(a \vee b)(k) = a(k) \vee b(k) \quad (k \in \partial_e K).$$

Suppose b_n converge in norm to $b \in A(K)$, where each b_n lies in the centre of $A(K)$. We show that b is in the centre of $A(K)$. If $a \in A(K)$ let $c_n \in A(K)$ with $c_n(k) = a(k)b_n(k)$ for all $k \in \partial_e K$. As functions in $A(K)$ attain their supremum on $\partial_e K$, we have

$$\|c_n - c_m\| = \|c_n - c_m\|_{\partial_e K} \leq \|b_n - b_m\| \|a\|,$$

and the right hand side tends to zero as $m, n \rightarrow \infty$. If

$c = \lim_n c_n$, then $c(k) = b(k)a(k)$ for all $k \in \partial_e K$, so that b is in the centre of $A(K)$.

If a, b belong to the centre of $A(K)$ we must show that $a \circ b \in A(K)$ with $(a \circ b)(k) = a(k)b(k)$ for all $k \in \partial_e K$ also lies in the centre of $A(K)$. If $p \in A(K)$ there is $p' \in A(K)$ with

$$p'(k) = b(k)p(k) \text{ for all } k \in \partial_e K.$$

But now there is $p'' \in A(K)$ with

$$p''(k) = a(k)p'(k) = a(k)b(k)p(k) = (a \circ b)(k)p(k)$$

for all $k \in \partial_e K$. Thus $a \circ b$ lies in the centre of $A(K)$.

The result about the order structure follows from the fact that $a \in A(K)^+$ if and only if $a|_{\partial_e K} \geq 0$, that $\|a\| = \|a\|_{\partial_e K}$ and Lemma IV.1.6 applied to the restriction of $\tilde{A}(K)$ to $\partial_e K$.

THEOREM IV.1.8. The map $a \mapsto a|_{\partial_e K}$ is a bijection from the centre of $A(K)$ onto the set of $\mathbb{Q}$ -continuous functions on $\partial_e K$.

We have only to show that if a belongs to the centre of $A(K)$ then $a|_{\partial_e K}$ is $\mathbb{Q}$ -continuous. It suffices to prove that $a|_{\partial_e K}$ is u.s.c., since $-a$ also lies in the centre of $A(K)$. I.e. to prove that for all $\alpha \in \mathbb{R}$

$$C_\alpha = \{k \in \partial_e K : a(k) \geq \alpha\} = F \cap \partial_e K$$

where F is a closed split face of K . Without loss of generality we assume $0 \leq a \leq 1$ and $0 \leq \alpha \leq 1$.

By Proposition IV.1.7 the function

$$a_n = [(a + (1 - \alpha)1_K) \wedge 1_K]^n$$

lies in the centre of $A(K)$ for $n = 1, 2, \dots$. As the operations are pointwise on $\partial_e K$,

$$\begin{aligned} \lim_n a_n(k) &= 1 \quad (k \in C_\alpha) \\ &= 0 \quad (k \in \partial_e K \setminus C_\alpha). \end{aligned}$$

Also (a_n) is decreasing on $\partial_e K$, and hence on K . Define

$b(k) = \inf a_n(k)$ for all $k \in K$, so that b is u.s.c. and affine,

whilst also $0 \leq b \leq 1$. Hence

$$F = \{k \in K : b(k) = 1\}$$

$$\text{and } G = \{k \in K : b(k) = 0\}$$

are faces, the former closed and the latter a G_δ . As $C_\alpha = F \cap \partial_e K$

it remains only to prove that F is split.

As c is u.s.c. and affine, and coincides with χ_F on $\partial_e K$, $c = \hat{\chi}_F$. By III.2.1 $F' = \hat{\chi}_F^{-1}(0) = G$, and by III.2.6 F is a parallel face. Suppose

$$\lambda x + (1 - \lambda)y = \lambda x' + (1 - \lambda)y' \quad (\mathfrak{E})$$

with $x, x' \in F$, $y, y' \in G$ and that $y \neq y'$. Let $f \in A(K)^+$ be such that $f(y) \neq f(y')$. Let $d_n(k) = a_n(k)f(k)$ ($k \in \partial_e K$) and $d_n \in A(K)^+$. Again (d_n) is decreasing on $\partial_e K$ and hence on K . Let $d = \inf d_n$ and again $d = (f\chi_F)|_{\partial_e K}^\wedge$, so that $d = f$ on F and $d = 0$ on G . Applying d to $(\mathfrak{E})$ we obtain $\lambda d(y) = \lambda d(y')$. Hence $f(y) = f(y')$ contrary to assumption. The proof is now complete.

COROLLARY IV.1.9. The centre of $A(K)$ is precisely $\tilde{A}(K)$.

An alternative characterisation of $A(K)$ is available to us. We define the ideal centre of $A(K)$ to be the set of all linear operators T on $A(K)$ which are order bounded, i.e. for which there exists $\lambda \in \mathbb{R}^+$ with $-\lambda I \leq T \leq \lambda I$, where I is the identity on $A(K)$. Trivially we have:

PROPOSITION IV.1.10. The ideal centre of $A(K)$ is an Archimedean ordered Banach space with order unit I .

We may thus endow the ideal centre of $A(K)$ with both the operator norm, $\| \cdot \|_{op}$, and the order unit norm induced by I , $\| \cdot \|_I$. In fact we shall see that these norms coincide.

THEOREM IV.1.11. The map $T \mapsto T1_K$ is an order preserving bijection of the ideal centre of $A(K)$ onto $\tilde{A}(K)$. Further for all T we have $\|T\|_{op} = \|T\|_I = \|T1_K\|$.

We regard $a \in A(K)$ as both an affine function on K and as a linear function on $A(K)^*$. We identify K with the state space of $A(K)$, noting that if $k \in \partial_e K$ then k lies on an extremal ray of K , i.e. if $0 \leq m \leq k$ then $m = \alpha k$ for some $\alpha \in \mathbb{R}^+$ (compare Proposition I.1.14). T^* will denote the adjoint operator on $A(K)^*$ which is also order bounded, as may be easily verified, and is positive if and only if T is.

To see that $T1_K \in \tilde{A}(K)$, suppose $a \in A(K)$. We claim that if $k \in \partial_e K$, $(T1_K)(k)a(k) = (Ta)(k)$, showing that $T1_K \in \tilde{A}(K)$. We may suppose T is positive by adding an appropriate multiple of I . Then $0 \leq T^*k \leq k$ if $k \geq 0$, so in particular if $k \in \partial_e K$ then $T^*k = \lambda k$. It is clear that $\lambda = \|T^*k\|$. Now if $k \in \partial_e K$

$$\begin{aligned} (T1_K)(k)a(k) &= 1_K(T^*k)a(k) \\ &= \|T^*k\| a(k) \\ &= a(\|T^*k\| k) \\ &= a(T^*k) \\ &= Ta(k), \end{aligned}$$

as claimed. As the map $f \mapsto T_f$ where $(T_f a)(k) = f(k)a(k)$ for all $k \in \partial_e K$ is obviously the inverse of $T \mapsto T1_K$ this latter map is onto.

It is clear that the map is order preserving. We prove the assertions about the norm. From definition of the operator norm we have

$$\|T\|_{op} \geq \|T1_K\|. \quad (\textcircled{a})$$

As $(T1_K)(k)a(k) = (Ta)(k)$ for all $k \in \partial_e K$ we have

$$|(Ta)(k)| \leq |a(k)| \|T1_K\| \text{ for all } k \in \partial_e K.$$

so that

$$\|Ta\| \leq \|T^1_K\| \|a\|.$$

I.e.

$$\|T\|_{op} \leq \|T^1_K\|. \quad (00)$$

If $a \geq 0$ from the definitions we have

$$- \|T^1_K\| a(k) \leq (Ta)(k) \leq \|T^1_K\| a(k) \text{ for all } k \in \partial_e K$$

so that

$$- \|T^1_K\| I \leq T \leq \|T^1_K\| I$$

and

$$\|T\|_I \leq \|T^1_K\|. \quad (000)$$

Finally,

$$- \|T\|_I I \leq T \leq \|T\|_I I$$

so

$$- \|T\|_{I^1_K} \leq T^1_K \leq \|T\|_{I^1_K}$$

and

$$\|T^1_K\| \leq \|T\|_I. \quad (0000)$$

Combining (0), (00), (000) and (0000) proves the equalities.

2 The split face topology for simplexes.

By II.3.7 every closed face of a simplex S is split, so in this case the $\mathbb{Q}$ -closed subsets of $\partial_e S$ are precisely the sets $F \cap \partial_e S$ for any closed face F of S . This should endow the centre of $A(S)$ with some extra properties, and it indeed does. The first thing that we note is that the $\mathbb{Q}$ -topology is T_1 (i.e. every singleton is closed). If S is a Bauer simplex then the $\mathbb{Q}$ -topology and the relative topology on $\partial_e S$ coincide. In fact now $\tilde{A}(S) = A(S)$. It is always true that $\tilde{A}(S) \neq \emptyset$ as the constants always belong to $\tilde{A}(S)$. However there need be no other members of $\tilde{A}(S)$, even in the case of a simplex. We are however able to provide alternative characterisations of $\tilde{A}(S)$.

THEOREM IV.2.1. Let S be a compact simplex, $a \in A(S)$. The following are equivalent:

- (1) $a|_{\partial_e S}$ is $\mathbb{Q}$ -continuous.
- (2) a is in the centre of $A(S)$.
- (3) a belongs to some subspace of $A(S)$ that is an algebra under the operation of multiplication pointwise on $\partial_e S$.
- (4) For all $\lambda \in \mathbb{R}$, the pair $(a, \lambda 1_S)$ has a supremum in $A(S)$.

$(1) \implies (2)$ is Theorem IV.1.3, $(2) \implies (3)$ by Proposition IV.1.7 and $(3) \implies (4)$ by noting that we may adjoin the constants to the subspace of (3) and then close it and it will remain an algebra. Now (4) follows by Lemma IV.1.6. We must prove $(4) \implies (1)$.

If $\alpha \in \mathbb{R}$, the set $\{k \in \partial_e S : a(k) \leq \alpha\}$ is equal (as the lattice operations are pointwise on $\partial_e S$) to $\{k \in \partial_e S : (a \vee \alpha)(k) = \alpha\} = \partial_e S \cap \{k \in S : (a \vee \alpha)(k) = \alpha\}$. Since the latter set is the set where $a \vee \alpha$ attains its infimum it is a closed face of S . Thus $a|_{\partial_e S}$ is $\mathbb{Q}$ -l.s.c. Also

$\{k \in \partial_e S : a(k) \geq \alpha\} = \{k \in \partial_e S : (a \vee \alpha)(k) = a(k)\} =$
 $\partial_e S \cap \{k \in S : (a \vee \alpha)(k) = a(k)\}$ and again the latter set is
a closed face since $a \vee \alpha \geq a$. Thus $a|_{\partial_e S}$ is $\mathbb{Q}$ -u.s.c. and
hence $\mathbb{Q}$ -continuous.

COROLLARY IV.2.2. Let S be a compact simplex, then

- (1) There is a largest lattice ordered subspace of $A(S)$ containing the constants.
- (2) There is a largest subspace of $A(S)$ that is an algebra for the operation of pointwise multiplication on $\partial_e S$.
Further both these subspaces coincide with $A(S)$.

We turn now to some properties of the $\mathbb{Q}$ -topology for metrisable simplexes. We first need some more information about $\mathbb{Q}$ -convergence. If $k \in S$, $Q(k)$ will denote the smallest closed face of S containing k .

PROPOSITION IV.2.3. If $k \in S$, $y \in \partial_e(Q(k))$ and (x_α) is a net in $\partial_e S$ converging to k , then (x_α) $\mathbb{Q}$ -converges to y .

If not there is an $\mathbb{Q}$ -closed subset D of $\partial_e S$ such that $x_\alpha \in D$ frequently whilst $y \notin D$. If $D = \partial_e S \cap F$ for F a closed face of S , then (x_α) is frequently in F so $k = \lim x_\alpha \in F$. Hence $y \in Q(k) \subset F$, and as y is extreme $y \in D$ - a contradiction.

A partial converse to this is provided by:

PROPOSITION IV.2.4. If (x_n) is a sequence in $\partial_e S$ converging to k , and $(x_{n'})$ is a subnet of (x_n) $\mathbb{Q}$ -converging to $y \in \partial_e S$, then $y \in Q(k)$.

If $y \notin Q(k)$ there exists N such that $n \geq N$ implies $x_n \neq y$. Let $R = \{x_n : n \geq N\} \cup Q(k)$, a closed subset of S . Since $\{x_n\} \subset \partial_e K$, R is dilated and hence $F = \overline{\text{co}}(R)$ is a face

of S , by II.3.2.

If $y \notin R$ then $y \notin F$ else y would be an extreme point of F and hence in R . I.e. $\partial_e S \setminus F$ is an $\mathbb{Q}$ -neighbourhood of y . Since (x_β) $\mathbb{Q}$ -converges to y , eventually $x_\beta \notin F$. But (x_β) is a subnet of (x_n) so is eventually in R , a contradiction.

Before proving the main result we have a lemma:

LEMMA IV.2.5. Let S be a metrisable simplex, and $R \subset \partial_e S$ be $\mathbb{Q}$ -compact. If $a \in A(S)$, $a \geq 0$ and $a(r) > 0$ for all $r \in R$, then there exists $\varepsilon > 0$ such that $a(r) > \varepsilon$ for all $r \in R$.

If not then for each n there is $r_n \in R$ with $a(r_n) < 1/n$. Choose a subsequence (r_m) converging to k in K , then $a(k) = \lim_m a(r_m) = 0$. Choose a subnet (r_β) of (r_m) which $\mathbb{Q}$ -converges to $y \in R$, possible as R is $\mathbb{Q}$ -compact. By IV.2.4 $y \in Q(k)$. Since $a(k) = 0$, $a|_{Q(k)} = 0$ (as $a \geq 0$) and hence $a(y) = 0$, a contradiction.

Suppose S is metrisable, then $A(S)$ is separable (I.1.17) so we may suppose $\{a_i : i = 1, 2, \dots\}$ to be a dense subset of $A(S)^+$. Define

$$U(i, n) = \{k \in \partial_e S : a_i(k) > 1/n\}.$$

THEOREM IV.2.6. Let S be a metrisable simplex and $k \in \partial_e S$. The following are equivalent:

- (1) k has a countable $\mathbb{Q}$ -neighbourhood base.
- (2) k has an $\mathbb{Q}$ -neighbourhood base of $\mathbb{Q}$ -compact sets.

(1) $\implies$ (2). Let Q be a closed face of S not containing k , then we must find an $\mathbb{Q}$ -compact $\mathbb{Q}$ -neighbourhood of k disjoint from Q . Choose $a \in A(K)^+$ such that $a|_Q = 0$, $a(k) \neq 0$, possible by Edwards' theorem. Let Q_n be an increasing sequence of

closed faces of S such that $\{\partial_e S \setminus Q_n\}$ is a basis of $\mathbb{Q}$ -neighbourhoods at k . We claim that for some N ,

$$\{y \in \partial_e S : a(y) < 1/N\} \subset Q_N. \quad (\text{EE})$$

If not, for all n there is $x_n \in \partial_e S$ such that $a(x_n) < 1/n$ and $x_n \notin Q_n$. Choose a subsequence (x_m) converging to $x \in K$. Then (x_m) $\mathbb{Q}$ -converges to k , so $k \in Q(x)$. Since a is continuous on K , $a(x) = 0$ and hence $a(k) = 0$, a contradiction.

Now $\{y \in \partial_e S : a(y) \geq 1/N\}$ is $\mathbb{Q}$ -compact, for if (D_γ) is a family of $\mathbb{Q}$ -closed subsets with the finite intersection property, let F_γ be closed faces of S with $D_\gamma = F_\gamma \cap \{y \in \partial_e S : a(y) \geq 1/N\}$. The family of all sets of the form $F_\gamma \cap \{y \in S : a(y) \geq 1/N\}$ has the finite intersection property so $\bigcap F_\gamma \cap \{y \in S : a(y) \geq 1/N\} \neq \emptyset$. As a attains its supremum on $\bigcap F_\gamma$ at an extreme point we have $\bigcap D_\gamma = \bigcap F_\gamma \cap \{y \in \partial_e S : a(y) \geq 1/N\} \neq \emptyset$. Also $\{y \in \partial_e S : a(y) \geq 1/N\}$ is an $\mathbb{Q}$ -neighbourhood of k , since it contains $\partial_e S \setminus Q_N$. As $a|_Q = 0$, this is the required $\mathbb{Q}$ -neighbourhood.

(2) $\implies$ (1). We claim that the set $\{U(i, n) : i, n \in \mathbb{Z}^+\}$ contains a (necessarily countable) neighbourhood base of k . If Q is a closed face of S not containing k , let $a \in A(S)^+$ with $a|_Q = 0$ and $a(k) \neq 0$. Let $Q' = \{x \in S : a(x) = 0\}$, a closed face of S containing Q , whilst $k \notin Q'$. Since $Q' \setminus \partial_e S$ is an $\mathbb{Q}$ -neighbourhood of k , there is an $\mathbb{Q}$ -compact $\mathbb{Q}$ -neighbourhood R of k such that $R \cap Q' = \emptyset$. Then $a|_R > 0$, so $a|_R > 2/n$ for some n by Lemma IV.2.5. Choose i so that $\|a_i - a\| < 1/n$, then $a_i|_R > 1/n$ so $R \subset U(i, n)$. Since R is an $\mathbb{Q}$ -neighbourhood of k so is $U(i, n)$. Also $a_i|_Q < 1/n$ so $Q \cap U(i, n) = \emptyset$ completing the proof.

We have in fact proved:

THEOREM IV.2.7. Let S be a metrisable simplex, then the following are equivalent:

- (1) The $\mathbb{Q}$ -topology is first countable.
- (2) The $\mathbb{Q}$ -topology is second countable.
- (3) The $\mathbb{Q}$ -topology is locally compact.

(2) $\Rightarrow$ (1) a fortiori.

(1) $\Rightarrow$ (3) by IV.2.6.

(3) $\Rightarrow$ (2) because under the assumption that every point of $\partial_e S$ has an $\mathbb{Q}$ -neighbourhood base of $\mathbb{Q}$ -compact sets the proof of (2) $\Rightarrow$ (1) of IV.2.6 shows that $\{U(i,n) : i, n = 1, 2, \dots\}$ contains an $\mathbb{Q}$ -neighbourhood base for the $\mathbb{Q}$ -topology on $\partial_e S$.

3 Parallel face topologies.

The characterisation of $A(S)$ given in Theorem IV.2.1 do not **remain** valid when S is replaced by an arbitrary compact convex set, as may be seen by considering, for example, a square. In this section ~~we~~ extend the notion of a facial topology to enable us to produce a result similar to IV.2.1. The role of split faces will be played by certain families of parallel faces (see the definition before III.2.6).

We define a family $\mathfrak{F}$ of parallel faces of a compact convex set K to be topological if it is closed under finite intersections, finite convex hulls, intersection of downward filtering families and if it contains $\emptyset$ and K .

We deal first with ways to construct topological families.

LEMMA IV.3.1. Let $\mathfrak{F}$ be a family of parallel faces of K , closed under finite intersections, finite convex hulls and containing $\emptyset$ and K . Let $\mathcal{S}(\mathfrak{F})$ be the family obtained by adjoining to $\mathfrak{F}$ the intersection of downward filtering families of $\mathfrak{F}$. Then $\mathcal{S}(\mathfrak{F})$ is a topological family of parallel faces of K .

That each member of $\mathcal{S}(\mathfrak{F})$ is a parallel face is because

$$\hat{\chi}_F = (\inf \chi_{F_\alpha})^\wedge = \inf (\chi_{F_\alpha})^\wedge$$

(by I.2.7) when $F = \bigcap F_\alpha$ for (F_α) a downward filtering family in $\mathfrak{F}$. By III.2.6 $(\chi_{F_\alpha})^\wedge$ is affine, so $(\chi_F)^\wedge$ is affine and F is a parallel face by III.2.6 again.

Suppose $F = \bigcap_A F_\alpha$, $G = \bigcap_B G_\beta$ are in $\mathcal{S}(\mathfrak{F})$. Then

$$F \cap G = \bigcap_{A \times B} F_\alpha \cap G_\beta \in \mathcal{S}(\mathfrak{F}).$$

If $M = \text{co}(F \cup G)$, then

$$N = \bigcap_{A \times B} \text{co}(E_\alpha \cup G_\beta) \in \mathcal{S}(\mathcal{F}) \quad \text{and } N \supset M.$$

Let $x \in N$, then for all α, β there is $y_{\alpha, \beta} \in F$, $z_{\alpha, \beta} \in G$ and $\lambda_{\alpha, \beta} \in [0, 1]$ such that $x = \lambda_{\alpha, \beta} y_{\alpha, \beta} + (1 - \lambda_{\alpha, \beta}) z_{\alpha, \beta}$. We may construct a directed set with terms in $A \times B$ such that $\lambda_{\alpha, \beta} \rightarrow \lambda \in [0, 1]$, $y_{\alpha, \beta} \rightarrow y \in F$ and $z_{\alpha, \beta} \rightarrow z \in G$, and then $x = \lambda y + (1 - \lambda) z$. Hence $N = M$, and $\mathcal{S}(\mathcal{F})$ is closed under finite convex hulls.

Finally let $F_\alpha \in \mathcal{S}(\mathcal{F})$ with F_α downward filtering. For each α , let $(F_\alpha^{i_\alpha})_{i_\alpha \in I_\alpha}$ be a downward filtering family from $\mathcal{F}$ with intersection F_α . The family $\left\{ \bigcap_{j=1}^n F_{\alpha_j}^{i_{\alpha_j}} \right\}$ is contained in $\mathcal{F}$, is downward filtering, and has intersection $\bigcap_\alpha F_\alpha$. I.e. $\bigcap_\alpha F_\alpha \in \mathcal{S}(\mathcal{F})$.

Finally that $\emptyset$ and K lie in $\mathcal{S}(\mathcal{F})$ is because they were given to lie in $\mathcal{F}$.

An obvious topological family of parallel faces is the family of all split closed faces of K . One way to construct new topological families is the following:

PROPOSITION IV.3.2. Let K, K' be compact convex sets and $\varphi : K \rightarrow K'$ be a continuous affine surjection such that $\varphi(\partial_e K) \subset \partial_e K'$. If $\mathcal{F}$ is a topological family of parallel faces of K' , then $\{\varphi^{-1}(F) : F \in \mathcal{F}\}$ is a topological family of parallel faces of K .

That $\varphi^{-1}(F)$ is a parallel face again uses III.2.6.

$$\begin{aligned} (\chi_{\varphi^{-1}(F)})^\wedge &= \inf \{ h : h \in A(K), h \geq \chi_{\varphi^{-1}(F)} \} \\ &\leq \inf \{ g \circ \varphi : g \in A(K'), g \geq \chi_F \} \\ &= (\chi_F)^\wedge \circ \varphi. \end{aligned}$$

Now $(\chi_{\varphi^{-1}(F)})^\wedge$ is u.s.c. and concave, $(\chi_F)^\wedge \circ \varphi$ is u.s.c. and affine. Both take value 1 on $\partial_e K \cap \varphi^{-1}(F)$ and 0 on

$\partial_e K \setminus \varphi^{-1}(F)$, so are equal by I.2.2. I.e. $(\chi_{\varphi^{-1}(F)})^\wedge$ is affine and $\varphi^{-1}(F)$ is a parallel face.

The intersection properties are trivial. We must prove that the family is closed under finite convex hulls. Let $F, G \in \mathcal{F}$ so that also $\text{co}(F \cup G) \in \mathcal{F}$. It is clear that $M = \text{co}(\varphi^{-1}(F) \cup \varphi^{-1}(G)) \subset \varphi^{-1}(\text{co}(F \cup G)) = N$. If $x \in \partial_e N$ then $x \in \partial_e K$ as N is a face of K . Hence $\varphi(x) \in \partial_e K' \cap \text{co}(F \cup G)$, so $\varphi(x) \in \partial_e F \cup \partial_e G$. Hence $x \in \varphi^{-1}(F) \cap \varphi^{-1}(G) \subset M$. By the Krein-Milman theorem it follows that $M = N$, so M belongs to the family.

The reason for introducing topological families is of course the following:

PROPOSITION IV.3.3. If $\mathcal{F}$ is a topological family of parallel faces of K then the sets $F \cap \partial_e K$ for $F \in \mathcal{F}$ are the closed sets for a compact topology on $\partial_e K$.

Such a topology will be termed a facial topology on $\partial_e K$. We can extend to this context Corollary IV.1.4.

THEOREM IV.3.4. Let τ be a facial topology on $\partial_e K$.

(1) Let $f : \partial_e K \rightarrow \mathbb{R}$ be a positive τ -u.s.c. function. Then f has a unique extension $f : K \rightarrow \mathbb{R}$ which is positive, u.s.c. and affine.

(2) Let $f : \partial_e K \rightarrow \mathbb{R}$ be τ -continuous. Then f has a unique extension $f \in A(K)$.

(1) Suppose $0 \leq f \leq 1$ and let n be a positive integer. The set

$$\{k \in \partial_e K : f(k) > 1/n\}$$

is τ -closed, so is of the form $\partial_e K \cap F_{i,n}$ for $F_{i,n}$ a parallel face. Define

$$\varphi_n = \sum_{i=1}^n (1/n) (\chi_{F_{i,n}})^\wedge.$$

Then φ_n is affine, u.s.c. and $|\varphi_n(k) - f(k)| < 1/n$ if $k \in \partial_e K$.

It follows that $\|\varphi_n - \varphi_m\| \leq 1/n + 1/m$ using I.3.2. Now φ_n converges uniformly on K to $\tilde{f}$ which is u.s.c. and affine. As it is clear that $\tilde{f}|_{\partial_e K} = f$ we are through. The uniqueness follows from I.2.3.

(2) Choose $a, b \in \mathbb{R}$ so that $f + a \geq 0$, $b - f \geq 0$. Then $u = (f + a)^\sim$ and $v = (b - f)^\sim$ are positive u.s.c. and affine. Now $u - a$ and $b - v$ are affine and respectively u.s.c. and l.s.c., and coincide with f on $\partial_e K$. Hence $\tilde{f} = u - a = b - v$ is continuous (I.2.2) and extends f . Again I.2.3 gives the uniqueness.

If τ is a facial topology on $\partial_e K$, let H_τ denote the set of all extensions to $A(K)$ of τ -continuous functions on $\partial_e K$. The following result is easily verified.

PROPOSITION IV.3.5. If τ is a facial topology on $\partial_e K$, then H_τ is a commutative Banach algebra and an M-space for the order and norm induced from $A(K)$, and multiplication $(a, b) \mapsto a \circ b$ where

$$(a \circ b)(k) = a(k)b(k) \text{ for all } k \in \partial_e K.$$

Further the lattice operations satisfy

$$(a \vee b)(k) = a(k) \vee b(k) \text{ for all } k \in \partial_e K.$$

We can use facial topologies to strengthen the equivalences of IV.1.5. The proof is the same, using Theorem IV.3.4 instead of Corollary IV.1.4.

PROPOSITION IV.3.6. The following conditions on a compact convex set K are equivalent:

- (1) K is a Bauer simplex.
- (2) There exists a Hausdorff facial topology on $\partial_e K$.

Let $\hat{A}(K)$ be the union of all the subspaces H_τ of $A(K)$, as τ ranges over all facial topologies on $\partial_e K$. We can use this set to extend the characterisations of Theorem IV.2.1 to the

non-simplicial case. We emphasise here that $\hat{A}(K)$ need not be a linear subspace of $A(K)$, as for example when K is a square.

THEOREM IV.3.7. Let K be a compact convex set and $a \in A(K)$, then the following are equivalent:

- (1) There is a facial topology τ on $\partial_e K$ such that $a|_{\partial_e K}$ is τ -continuous.
- (2) There is a subspace of $A(K)$ containing a which is an algebra for the operation of pointwise multiplication on $\partial_e K$.

- (3) For all finite families of pairs of reals (α_i, β_i) , $\bigvee_{i=1}^n (\alpha_i a + \beta_i)$ exists in $A(K)$.

(1) $\implies$ (2) is IV.3.5 and (2) $\implies$ (3) follows from Lemma IV.1.6 after adjoining the constants to the subspace and closing it.

(3) $\implies$ (1). The set Γ of all functions of the form $\bigvee_{i=1}^n (\alpha_i a + \beta_i)$ is a cone that is closed under suprema. By Lemma I.1.16 $\Gamma - \Gamma$ is a sublattice of $C(K)$ containing the constants and a . The same will be true for $W = \overline{\Gamma - \Gamma}$. We may identify W with $A(S(W))$ and as W is a lattice $S(W)$ is a Bauer simplex and W is isometrically order isomorphic to $C(\mathcal{N})$ where $\mathcal{N} = \partial_e(S(W))$.

Let I be the natural injection of $C(\mathcal{N})$ into $A(K)$, then the associated continuous affine function $\pi : K \longrightarrow P(\mathcal{N})$ is easily seen to be onto and extreme point preserving. By Proposition IV.3.2 the family $\{\pi^{-1}(F) : F \text{ is a closed face of } P(\mathcal{N})\}$ is a topological family of parallel faces of K . The functions of W are easily seen to be continuous for the associated facial topology, and hence (1) is true since $a \in W$.

COROLLARY IV.3.8. Let K be a compact convex set, then the following three sets coincide:

- (1) $\hat{A}(K)$.
- (2) The union of all the subspaces of $A(K)$ which are algebras for the operation of pointwise multiplication on $\partial_e K$.
- (3) The union of all the vector lattices contained in $A(K)$ which contain the constants.

4 Hull-kernel topology.

For a class of compact convex sets, the $\mathcal{Q}$ -topology on $\partial_e K$ is closely related to a "hull-kernel" topology that is similar to that for algebras. If $k \in \partial_e K$ we let F_k denote the smallest closed split face containing k . We first study a "structure topology" on the family $\{F_k : k \in \partial_e K\}$. We consider compact convex sets which satisfy Størmer's axiom:

If $\{F_\alpha : \alpha \in A\}$ is a family of closed split faces of K then $\overline{\text{co}}(\bigcup_\alpha F_\alpha)$ is a closed split face of K .

PROPOSITION IV.4.1. Let K satisfy Størmer's axiom. The set $\mathcal{E} = \{F_k : k \in \partial_e K\}$ can be topologised by the closure operation $S \mapsto \bar{S}$ where $\bar{S} = \{F \in \mathcal{E} : F \subset \overline{\text{co}}\{G : G \in S\}\}$.

The map $k \mapsto F_k$ is continuous and open from the $\mathcal{Q}$ -topology of $\partial_e K$ to this topology of $\mathcal{E}$. Moreover for this topology $\mathcal{E}$ is T_0 .

Clearly $\bar{\emptyset} = \emptyset$, $\bar{\mathcal{E}} = \mathcal{E}$ and $\bar{S} \supset S$ for all $S \subset \mathcal{E}$. If $H \in \bar{S}$ then by definition $H \subset \overline{\text{co}} \bigcup_{G \in S} G$, so that $\overline{\text{co}} \bigcup_{H \in \bar{S}} H \subset \overline{\text{co}} \bigcup_{G \in S} G$, and hence $\bar{\bar{S}} = \bar{S}$. It remains to prove that $\overline{S \cup T} = \bar{S} \cup \bar{T}$. That $\bar{S} \cup \bar{T} \subset \overline{S \cup T}$ is trivial. Suppose $F_k \in \overline{S \cup T}$ where $k \in \partial_e K$. Then $F_k \subset \overline{\text{co}} \bigcup_{G \in S \cup T} G$ and so $k \in \overline{\text{co}} \bigcup_{G \in S \cup T} G$. Since $k \in \partial_e K$ we can apply Milman's theorem to obtain $k \in \overline{\bigcup_{G \in S \cup T} G} \subset \overline{\bigcup_{G \in S} G} \cup \overline{\bigcup_{G \in T} G}$. By symmetry we may assume $k \in \overline{\bigcup_{G \in S} G} \subset \overline{\text{co}}(\bigcup_{G \in S} G)$. By Størmer's axiom the right hand term of this last relation is a closed split face. Hence $F_k \subset \overline{\text{co}} \bigcup_{G \in S} G = \bar{S}$. I.e. $\overline{S \cup T} = \bar{S} \cup \bar{T}$, and it follows that $S \mapsto \bar{S}$ is a topological closure operation.

Let S be a closed subset of $\mathcal{E}$ and consider the set $\hat{S} = \{k \in \partial_e K : F_k \in S\}$. We shall have

$$F_k \in S \iff F_k \subset \overline{\text{co}} \bigcup_{G \in S} G \iff k \in \overline{\text{co}} \bigcup_{G \in S} G,$$

where the last equivalence follows by Størmer's axiom. Now

$\hat{S} = \partial_e K \cap \overline{\text{co}} \bigcup_{G \in S} G$ is closed in $\partial_e K$ for the $\mathcal{Q}$ -topology. I.e.

we have proved that $k \mapsto F_k$ is continuous.

To prove that $k \mapsto F_k$ is open, suppose $U \subset \partial_e K$ is $\mathcal{Q}$ -open, so that $U = \partial_e K \setminus F$ where F is a closed split face of K . We must show that $\{F_k : k \in U\}$ is an open subset of $\mathcal{E}$.

Suppose $H \in \mathcal{E} \setminus \{F_k : k \in U\}$, then $H = F_x$ for some $x \in \partial_e K$, and we must have $x \in F$ for else $x \in U$ and then $H = F_x \in \{F_k : k \in U\}$. Hence $H = F_x \subset F$ so we have shown that $F \supset \overline{\text{co}} \bigcup \{H : H \in \mathcal{E} \setminus \{F_k : k \in U\}\}$.

If $k \in U$ then $k \notin \overline{\text{co}} \bigcup \{H : H \in \mathcal{E} \setminus \{F_k : k \in U\}\}$ because of the last inclusion. It follows that $F_k \notin \overline{\text{co}} \bigcup \{H : H \in \mathcal{E} \setminus \{F_k : k \in U\}\}$ whenever $k \in U$. By definition of the topology on $\mathcal{E}$ this suffices to prove that $\{F_k : k \in U\}$ is open.

Finally let F, G belong to $\mathcal{E}$ and suppose $F \not\subset G$. Then $\overline{\{G\}} = \{H \in \mathcal{E} : H \subset G\}$ so that $F \notin \overline{\{G\}}$. Hence $\mathcal{E} \setminus \overline{\{G\}}$ is an open neighbourhood of F not containing G , which shows that the topology is T_0 .

This topology we term the structure topology on $\mathcal{E}$.

A continuous affine bijection of K which leaves invariant all the faces $\{F_k : k \in \partial_e K\}$ is termed an inner automorphism of K . The set of all such will be denoted by U . We say that K admits sufficiently many inner automorphisms if for each $k \in \partial_e K$,

$$F_k = \overline{\text{co}} \{ \alpha(k) : \alpha \in U \}.$$

PROPOSITION IV.4.2. If K is a compact convex set satisfying Størmer's axiom and admitting sufficiently many inner automorphisms, then $k \mapsto F_k$ is open from $\partial_e K$ with the relative topology onto $\mathcal{E}$ with the structure topology.

Let V be open in K , and we shall show that $\{F_k : k \in V \cap \partial_e K\}$ is structurally open in $\mathcal{E}$. If this is not so, suppose

there is $G = F_{k_0}$ for $k_0 \in V \cap \partial_e K$ such that $G \subset \overline{\text{co}} \cup \{H : H \in \mathcal{E} \setminus \{F_k : k \in V \cap \partial_e K\}\}$. We may transform this to

$$G \subset \overline{\text{co}}(\partial_e K \cap \cup \{H : H \in \mathcal{E} \setminus \{F_k : k \in V \cap \partial_e K\}\}),$$

and then by Milman's theorem $k_0 \in \overline{\partial_e K \cap \cup \{H : H \in \mathcal{E} \setminus \{F_k : k \in V \cap \partial_e K\}\}}$.

As V is an open neighbourhood of k_0 in K , there is thus k_1 such that

$$k_1 \in V \cap \partial_e K \cap \cup \{H : H \in \mathcal{E} \setminus \{F_k : k \in V \cap \partial_e K\}\}.$$

We now choose $x \in \partial_e K$ such that $k_1 \in V \cap \partial_e K \cap F_x$ and such that $F_x \neq F_k$ for all $k \in V \cap \partial_e K$.

It follows from the existence of sufficiently many inner automorphisms that

$$k_1 \in \overline{\text{co}} \{ \alpha(x) : \alpha \in U \},$$

and by Milman's theorem

$$k_1 \in \overline{\{ \alpha(x) : \alpha \in U \}}.$$

Since V is an open neighbourhood of k_1 , there is $y \in V \cap \{ \alpha(x) : \alpha \in U \}$. Now $y \in \partial_e K$ since every $\alpha \in U$ maps $\partial_e K$ onto itself, so $y \in V \cap \partial_e K$ and hence $x \in F_y$. This clearly implies that $F_x = F_y$, which is a contradiction, so the proof is complete.

We may apply these last two results to obtain a characterisation of Bauer simplexes amongst all simplexes promised in Chapter I.

THEOREM IV.4.3. If S is a simplex the following are equivalent:

- (1) S is a Bauer simplex.
- (2) If F is a face of S , then so is $\bar{F}$.
- (3) S satisfies Størmer's axiom.

(1) $\implies$ (2) is Theorem II.2.6.

(2) $\implies$ (3). Let $\{F_\alpha\}$ be a family of closed faces of S (automatically split by II.3.5), we must show that $\overline{\text{co}(\bigcup F_\alpha)}$ is a face of S . By (2) it will suffice to prove that $\text{co}(\bigcup F_\alpha)$ is a face of S .

If $k = \sum_{i=1}^n \lambda_i k_i \in \text{co}(\bigcup F_\alpha)$, where $\sum_{i=1}^n \lambda_i = 1$, $\lambda_i > 0$ and $k_i \in F_{\alpha_i}$ for $i = 1, 2, \dots, n$, let $G = \text{co}(F_{\alpha_1} \dots F_{\alpha_n})$ so that $k \in G$. By III.2.4 G is a (split) face of S so that if $k = \mu x + (1 - \mu)y$ where $x, y \in S$, $0 < \mu < 1$ then $x, y \in G \subset \text{co}(\bigcup F_\alpha)$. It follows that the latter set is a face as required.

(3) $\implies$ (1). As S is a simplex, $F_k = \{k\}$ whenever $k \in \partial_e S$. It follows that the map $k \mapsto F_k$ is a homeomorphism of $\partial_e S$ onto $\mathcal{E}$ when $\mathcal{E}$ has the structure topology, and $\partial_e S$ has the $\mathbb{Q}$ -topology (IV.4.1) or the relative topology (IV.4.2, the existence of sufficiently many inner automorphisms being trivial). It follows that the $\mathbb{Q}$ -topology and the relative topology on $\partial_e S$ coincide. Now S is a Bauer simplex by Theorem IV.1.5.

We turn now to the hull-kernel topology. We call an ideal J in $A(K)$ fixed at $k \in \partial_e K$ if $a(k) = 0$ for all $a \in J$. By the duality of near lattice ideals and closed split faces of K , we have:

PROPOSITION IV.4.4. If $k \in \partial_e K$ there exists a largest near lattice ideal J_k which is fixed at k . In fact $J_k = \{a \in A(K) : \underline{a|_{F_k}} = 0\}$.

We emphasise that J_k is only maximal amongst near lattice ideals fixed at k , and is not necessarily maximal. Such an ideal we term a primitive ideal. The family $\{J_k : k \in \partial_e K\}$ is termed the primitive ideal space of $A(K)$, $\text{Prim}(A(K))$.

LEMMA IV.4.5. If K satisfies Störmer's axiom, and $\{F_\alpha\}$ is a family of closed split faces of K, then

$$\overline{\text{co}}(\bigcup_{\alpha} F_{\alpha}) = (\bigcap_{\alpha} F_{\alpha}^{\perp})^{\perp},$$

$$(\overline{\text{co}}(\bigcup_{\alpha} F_{\alpha}))^{\perp} = \bigcap_{\alpha} F_{\alpha}^{\perp}.$$

It is easily verified that $\overline{\text{co}}(\bigcup_{\alpha} F_{\alpha}) \subset (\bigcap_{\alpha} F_{\alpha}^{\perp})^{\perp}$. To prove the reverse inclusion, let $k \in K \setminus \overline{\text{co}}(\bigcup_{\alpha} F_{\alpha})$. By Hahn-Banach separation there is $a \in A(K)$ with $a(k) > 0$ and $a < 0$ on $\overline{\text{co}}(\bigcup_{\alpha} F_{\alpha})$. By Theorem III.2.11 there is $b \in A(K)$ such that $b \geq a$ and $b|_{\overline{\text{co}}(\bigcup_{\alpha} F_{\alpha})} = 0$. Now $b \in \bigcap_{\alpha} F_{\alpha}^{\perp}$ and $b(k) > 0$ so that $k \notin (\bigcap_{\alpha} F_{\alpha}^{\perp})^{\perp}$. Hence $\overline{\text{co}}(\bigcup_{\alpha} F_{\alpha}) = (\bigcap_{\alpha} F_{\alpha}^{\perp})^{\perp}$. The verification of the second inequality is now trivial.

If $\mathcal{J} \subset \text{Prim}(A(K))$ we define its kernel to be the set $k(\mathcal{J}) \subset A(K)$,

$$k(\mathcal{J}) = \bigcap_{J \in \mathcal{J}} J.$$

If K satisfies Störmer's axiom then Lemma IV.4.5 tells us that $k(\mathcal{J})$ is a near lattice ideal of $A(K)$. If I is a near lattice ideal of $A(K)$ its hull, $h(I)$, is defined by

$$h(I) = \{J \in \text{Prim}(A(K)) : I \subset J\}.$$

PROPOSITION IV.4.6. If K satisfies Störmer's axiom the operation $\mathcal{J} \mapsto h(k(\mathcal{J}))$ is a closure operation on $\text{Prim}(A(K))$.

$F \mapsto F^{\perp}$ is a homeomorphism of $\mathcal{E}$ with the structure topology onto $\text{Prim}(A(K))$ with this topology.

If $\mathcal{S} \subset \mathcal{E}$, define $\mathcal{J} = \{F^{\perp} : F \in \mathcal{S}\}$. By Lemma IV.4.5 we shall have

$$\overline{\text{co}}(\bigcup_{F \in \mathcal{S}} F) = (\bigcap_{F \in \mathcal{S}} F^{\perp})^{\perp} = (\bigcap_{J \in \mathcal{J}} J)^{\perp} = k(\mathcal{J})^{\perp}.$$

Hence we have the following equivalences if $G \in \mathcal{E}$:

$$G \in \overline{\mathcal{S}} \iff G \subset \overline{\text{co}}(\bigcup_{F \in \mathcal{S}} F)$$

$$\iff G \subset k(\mathcal{J})^{\perp}$$

$$\begin{aligned} \iff k(\mathfrak{J}) \subset G^\perp \\ \iff G^\perp \in h(k(\mathfrak{J})). \end{aligned}$$

This shows the desired homeomorphism.

We call this topology the hull-kernel topology on $\text{Prim}(A(K))$. Combining this last result with Propositions IV.4.1 and IV.4.2 we obtain:

THEOREM IV.4.7. If K satisfies Størmer's axiom then the hull-kernel topology of $\text{Prim}(A(K))$ is T_0 , and the map $k \mapsto J_k$ is continuous and open from the $\mathcal{Q}$ -topology of $\partial_e K$ to the hull-kernel topology of $\text{Prim}(A(K))$.

THEOREM IV.4.8. If K is a compact convex set satisfying Størmer's axiom and admitting sufficiently many inner automorphisms, then the map $k \mapsto J_k$ is continuous and open from the relative topology of $\partial_e K$ to the hull-kernel topology of $\text{Prim}(A(K))$.

Hence $\text{Prim}(A(K))$ is a Baire space in the hull-kernel topology.

The last remark follows from the fact that $\partial_e K$ is a Baire space for the relative topology (I.1.4).

CHAPTER V

MORPHISMS OF COMPACT CONVEX SETS

1 General morphisms.

If K_1 and K_2 are compact convex sets, we term a continuous affine function $\pi : K_1 \rightarrow K_2$ a morphism of K_1 into K_2 . We look first at the possibility of extending functions defined on the set of extreme points of K_1 with values in K_2 to a morphism of K_1 into K_2 . We deal first with the case that K_2 is a subset of the reals.

Let S be a cone in $C(X)$, for X a compact Hausdorff space, containing the constants, separating the points of X and closed under finite suprema. The upper S -envelope of f is defined by

$$\hat{f}_S(x) = \inf \{g(x) : g \geq f, g \in S\}.$$

Define a pre-order on $M(X)$ by $\mu <_S \nu$ if and only if $\mu(f) \leq \nu(f)$ for all $f \in S$. The following result may be proved similarly to I.3.8.

PROPOSITION V.1.1. If S is such a cone and $\mu \in M^+(X)$ the following are equivalent:

- (1) μ is $<_S$ -maximal.
- (2) $\mu(\hat{f}_S) = \mu(f)$ for all $f \in C(X)$.
- (3) $\mu(\hat{f}_S) = \mu(f)$ for all $f \in S$.

Proved similarly to I.3.10 is:

PROPOSITION V.1.2. If S is such a cone and $\nu \in M(X)$ there is a $<_S$ -maximal $\mu \in M(X)$ with $\mu \geq_S \nu$.

Finally before the main result we have:

LEMMA V.1.3. Let K be a compact convex set and $f : \partial_e K \rightarrow \mathbb{R}$ be such that $\hat{f}$ and $\check{f}$ coincide on $X = \overline{\partial_e K}$. Let $A = \{a|_X : a \in A(K)\}$ and $\tilde{A}$ be the smallest cone closed under finite suprema containing A . Similarly let B be the linear span of A and $\bar{f} = \hat{f}|_X = \check{f}|_X$, and take $\tilde{B}$ to be the smallest cone containing B and closed under finite suprema. Now

$$\hat{g}_{\tilde{A}} = \hat{g}_{\tilde{B}} \text{ for all } g \in C(X).$$

Clearly $\hat{g}_{\tilde{B}} \leq \hat{g}_{\tilde{A}}$. To prove the converse suppose $x \in X$ and $\varepsilon > 0$. Now there is $a_0 \in A(K)$ and $\lambda \in \mathbb{R}$ such that

$$g \leq a_0|_X + \lambda \bar{f}, \quad a_0(x) + \lambda \bar{f}(x) < \hat{g}_{\tilde{B}}(x) + \varepsilon/2.$$

There are three possibilities: if $\lambda > 0$ choose $a_1 \in A(K)$ with

$$\bar{f} = \hat{f}|_X \leq a_1|_X, \quad a_1(x) < \bar{f}(x) + \varepsilon/2\lambda.$$

If $\lambda < 0$ choose $a_1 \in A(K)$ with

$$a_1|_X \leq \check{f}|_X = \bar{f}, \quad \bar{f}(x) + \varepsilon/2\lambda < a_1(x).$$

If $\lambda = 0$ any choice of a_1 will do. In any case we have

$$g \leq (a_0 + \lambda a_1)|_X, \quad a_0(x) + \lambda a_1(x) < \hat{g}_{\tilde{B}}(x) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this means that $\hat{g}_{\tilde{A}} \leq \hat{g}_{\tilde{B}}$ and the lemma is proved.

It follows that the orderings $\angle_{\tilde{A}}$ and $\angle_{\tilde{B}}$ coincide on $M(X)$. Furthermore as $\tilde{A}$ is dense in the set of restrictions to X of continuous convex functions on K , both these orderings coincide with the usual order on $M(X)$, $\angle$, when this is considered as a subspace of $M(K)$.

THEOREM V.1.4. Let K be a compact convex set and $f : \partial_e K \rightarrow \mathbb{R}$ be a continuous function. f can be extended to an element of $A(K)$ if and only if the following two conditions are satisfied:

$$(1) \quad \hat{f}|_{\overline{\partial_e K}} = \check{f}|_{\overline{\partial_e K}}.$$

(2) The common restriction of $\hat{f}$ and $\check{f}$ to $\overline{\partial_e K}$ is annihilated by every boundary affine dependence.

The necessity is obvious. To prove sufficiency we adopt the notation of lemma V.1.3, and note that we must prove that $\bar{f} \in A$. For this it will suffice to prove that if $\mu \in M(X)$ and $\mu(a) = 0$ for all $a \in A$ then $\mu(\bar{f}) = 0$, so we take μ to be such a measure. Let $\mu = \alpha_1 \mu_1 - \alpha_2 \mu_2$ where $\alpha_1, \alpha_2 > 0$ and $\mu_1, \mu_2 \in P(X)$. Let ν_i be maximal measures on X with $\nu_i \succ \mu_i$, which is possible by V.1.2 and the remark before the proof. If $k \in B$ then again using the coincidence of the three orderings, $\mu_i(k) \leq \nu_i(k)$ and $\mu_i(-k) \leq \nu_i(-k)$, so that $\nu_i(\bar{f}) = \mu_i(\bar{f})$ and $\nu_i(a) = \mu_i(a)$ if $a \in A$.

By assumption $(\alpha_1 \mu_1 - \alpha_2 \mu_2)(a) = 0$ for all $a \in A$ so also $(\alpha_1 \nu_1 - \alpha_2 \nu_2)(a) = 0$ for all $a \in A$. As $(\alpha_1 \nu_1 - \alpha_2 \nu_2)$ is a boundary measure, hypothesis (2) tells us that $(\alpha_1 \nu_1 - \alpha_2 \nu_2)(\bar{f}) = 0$. Hence $\mu(\bar{f}) = 0$ as required, and the proof is complete.

COROLLARY V.1.5. Let K_1 and K_2 be compact convex sets, let $\varphi : \partial_e K_1 \rightarrow K_2$ be continuous and assume that $(a \circ \varphi)^\wedge|_{\overline{\partial_e K_1}} = (a \circ \varphi)^\vee|_{\overline{\partial_e K_1}}$ for all $a \in A(K_2)$. Then φ is uniformly continuous with a continuous extension $\bar{\varphi} : \overline{\partial_e K_1} \rightarrow K_2$. If whenever ν is a boundary affine dependence on K_1 , $\bar{\varphi} \nu$ is an affine dependence on K_2 , then φ can be extended to a morphism of K_1 into K_2 .

If $a \in A(K_2)$ the common restriction of $(a \circ \varphi)^\wedge$ and $(a \circ \varphi)^\vee$ to $\overline{\partial_e K_1}$ is continuous, hence uniformly continuous. Hence $a \circ \varphi = (a \circ \varphi)^\wedge|_{\partial_e K_1} = (a \circ \varphi)^\vee|_{\partial_e K_1}$ is uniformly continuous on $\partial_e K_1$ for every $a \in A(K)$. By compactness the weak topology on K_2 (defined by $A(K_2)$) coincides with the given topology.

Hence $\varphi : \partial_e K_1 \longrightarrow K_2$ is uniformly continuous.

Let $a \in A(K_2)$, then $(a \circ \varphi)^\wedge \Big|_{\overline{\partial_e K_1}} = (a \circ \varphi)^\vee \Big|_{\overline{\partial_e K_1}} = a \circ \bar{\varphi}$, so for every boundary affine dependence ν on K_1 ,

$$\int (a \circ \bar{\varphi}) d\nu = \int a d(\bar{\varphi} \nu) = 0.$$

Hence by Theorem V.1.4 there is $a' \in A(K_1)$ with $a' \Big|_{\overline{\partial_e K_1}} = a \circ \bar{\varphi}$.

If $k \in K_1$, let μ, μ' be boundary measures representing k , and let z, z' be the resultants of $\bar{\varphi} \mu, \bar{\varphi} \mu'$ respectively. We claim $z = z'$, for $\mu - \mu'$ is a boundary affine dependence on K_1 , hence $\bar{\varphi} \mu - \bar{\varphi} \mu' = \bar{\varphi}(\mu - \mu')$ is an affine dependence on K_2 , so $\bar{\varphi} \mu$ and $\bar{\varphi} \mu'$ have the same resultant. Define $\tilde{\varphi} : K_1 \longrightarrow K_2$ by $\tilde{\varphi}(k)$ is the resultant of $\bar{\varphi} \mu$ where μ is a boundary measure representing k . Clearly $\tilde{\varphi} \Big|_{\partial_e K_1} = \varphi$ for if $k \in \partial_e K_1$ the only measure representing k is ε_k .

We must show that $\tilde{\varphi}$ is continuous, it being clearly affine. We retain the above notation.

$a'(k) = \int a' d\mu = \int (a \circ \bar{\varphi}) d\mu = \int a d(\bar{\varphi} \mu) = a(z) = a(\tilde{\varphi}(k))$ whenever $a \in A(K_2)$. Hence $a' = a \circ \varphi$ so $\tilde{\varphi}$ is continuous from the given topology of K_1 to the weak topology of K_2 , which again coincides with that given. Hence $\tilde{\varphi}$ is the required extension.

We turn now to projective compact convex sets. We refer the reader to [39] for a definition of a category. We consider here the category whose objects are compact convex sets and whose morphisms are continuous affine maps.

PROPOSITION V.1.6. A compact convex set K is projective if and only if whenever X and Y are compact convex sets

$\pi : X \longrightarrow Y$ is a surjective morphism and $\varphi : K \longrightarrow Y$ is any

morphism, then there is a morphism $\psi : K \longrightarrow X$ such that $\varphi = \pi \circ \psi$.

The reader may either take this to be a (trivial) result using the definition of projectivity given in [39] or as a definition in itself. This will not affect later work. We also need the following definition: a topological space is extremally disconnected if the closure of every open set is open. A compact Hausdorff space which is extremally disconnected we shall term Stonian.

PROPOSITION V.1.7. Let Ω be a compact Hausdorff space. The vector lattice of all continuous functions on Ω is complete if and only if Ω is Stonian.

Suppose $C(\Omega)$ is complete, and let G be a proper open subset of Ω . Let F be the set of all continuous functions on Ω with values in $[0,1]$ that take value 0 off G . The family is bounded above by the constantly one function, so has a supremum, g . If $w \in G$ there is a function $f \in F$ with $f(w) = 1$, by Urysohn's lemma. Hence $g(w) = 1$ and since g is a continuous function $g|_{\bar{G}} = 1$. If $w_1 \notin \bar{G}$ then there is a continuous function h on Ω with values in $[0,1]$ such that $h(w_1) = 0$ and $h = 1$ on G . Thus h dominates all the functions in F , so $h \geq g$. Hence $g(w_1) = 0$, and we see that g is the characteristic function of $\bar{G}$. As g is continuous, $\bar{G}$ is open.

Suppose conversely that Ω is Stonian. It will suffice to show that every family F of non-negative functions in $C(\Omega)$ has an infimum. For each $\lambda > 0$ let

$$G_\lambda = \{w : f(w) < \lambda \text{ for some } f \in F\}.$$

We see that G_λ is open, $G_\lambda \subset G_\mu$ is $\lambda \leq \mu$ and $\bigcup \{G_\lambda : \lambda > 0\} = \Omega$. For each $w \in \Omega$, we can therefore define

$$g(w) = \inf \{\lambda > 0 : w \in \bar{G}_\lambda\}.$$

If $\mu > 0$ then

$$\{w : g(w) < \mu\} = \bigcup \{\bar{G}_\lambda : \lambda < \mu\},$$

which is open, since $\mathcal{N}$ is extremally disconnected. If $\mu \geq 0$ then

$$\{\omega : g(\omega) \leq \mu\} = \bigcap \{\overline{G_\lambda} : \lambda > \mu\},$$

which is closed. It follows that g is continuous.

If $\lambda < g(\omega)$ then $\omega \notin G_\lambda$ so $f(\omega) \geq \lambda$ for all $f \in F$.

Hence $g \leq f$ for all $f \in F$. Suppose $h \in C(\mathcal{N})$ and $h \leq f$ for all

$f \in F$. If $\omega \in G_\lambda$ then $h(\omega) < \lambda$. Since h is continuous,

$\overline{G_\lambda} \subset \{\omega : h(\omega) \leq \lambda\}$. If $g(\omega) < \lambda$ then $\omega \in \overline{G_\lambda}$, so $h(\omega) \leq \lambda$.

Hence $h \leq g$, so g is the infimum of the family F as required.

THEOREM V.1.8. In the category of compact convex sets and continuous affine maps the projective sets are the Bauer simplexes with extremally disconnected extreme point set.

In order to prove this result it will be convenient to work with the affine spaces. K is projective if and only if for all compact convex sets X and Y and for all positive, order unit preserving linear operators $T : A(Y) \rightarrow A(X)$ and $S : A(X) \rightarrow A(K)$, with $T(1,1)$, there is a positive unit preserving $U : A(X) \rightarrow A(K)$ such that $U \circ T = S$. If K is a Bauer simplex with extremally disconnected extreme boundary, then $A(K)$ is a complete vector lattice by II.2.3 and V.1.7. The existence of U is now guaranteed by the monotone extension theorem.

Suppose conversely that K has this property. We take $A(X)$ to be $\mathcal{C}^\infty(K)$, $A(Y) = A(K)$, S the identity on $A(K)$ and T the natural embedding of $A(K)$ into $A(X)$. Suppose $U : A(X) \rightarrow A(K)$ is positive and unit preserving with $U \circ T = S$. Suppose $\{f_\alpha : \alpha \in A\} \subset A(K)$ and that $h \in A(K)$ with $h \geq f_\alpha$ for all $\alpha \in A$. Let g be the supremum in $A(X)$ of the family $\{Tf_\alpha : \alpha \in A\}$ which is bounded above by Th . We claim Ug is the supremum of the family $\{f_\alpha : \alpha \in A\}$. We certainly have $Ug \geq U(Tf_\alpha) = f_\alpha$

for all $\alpha \in A$. On the other hand if $g' \geq f_\alpha$ for all $\alpha \in A$, then $Tg' \geq Tf_\alpha$ for all $\alpha \in A$, so $Tg' \geq g$. It follows that $g' = U(Tg') \geq Ug$. Thus Ug is the required supremum. We have thus proved that $A(K)$ is a complete vector lattice, so K is a Bauer simplex, by II.2.3, and with extremally disconnected set of extreme points, by V.1.7.

2 Extreme morphisms.

Let K_1, K_2 be compact convex sets, and let $A(K_1, K_2)$ denote the convex set of continuous affine maps of K_1 into K_2 . An obvious question of interest is the nature of the extreme points of this set. We shall limit ourself to simplexes. In fact we shall need even more limitations in order to obtain results. In the case that both sets are Bauer simplexes we have a very nice characterisation of the extreme points. First we need:

LEMMA V.2.1. Let V_1, V_2 be vector lattices and $T_1, T_2 : V_1 \longrightarrow V_2$ be positive linear operators. If $T_1 + T_2$ is a lattice homomorphism then so are T_1 and T_2 .

Let $a, b \in V_1$, then if $T_1 + T_2$ is a lattice homomorphism we have:

$$T_1(a \wedge b) + T_2(a \wedge b) = [T_1(a) + T_2(a)] \wedge [T_1(b) + T_2(b)].$$

It is clear that

$$T_1(a) \wedge T_1(b) + T_2(a) \wedge T_2(b) \leq [T_1(a) + T_2(a)] \wedge [T_1(b) + T_2(b)],$$

and since T_1 and T_2 are positive,

$$T_1(a) \wedge T_1(b) + T_2(a) \wedge T_2(b) \geq T_1(a \wedge b) + T_2(a \wedge b).$$

Hence

$$T_1(a) \wedge T_1(b) + T_2(a) \wedge T_2(b) = T_1(a \wedge b) + T_2(a \wedge b).$$

Thus

$$\begin{aligned} T_1(a \wedge b) &= T_1(a) \wedge T_1(b) + T_2(a) \wedge T_2(b) - T_2(a \wedge b) \\ &\geq T_1(a) \wedge T_1(b) \\ &\geq T_1(a \wedge b). \end{aligned}$$

I.e. $T_1(a \wedge b) = T_1(a) \wedge T_1(b)$ and it follows that T_1 is a lattice homomorphism. A similar proof shows that this holds also for T_2 .

THEOREM V.2.2. Let $\mathcal{N}_1, \mathcal{N}_2$ be compact Hausdorff spaces and $\pi \in A(P(\mathcal{N}_1), P(\mathcal{N}_2))$. The following are equivalent:

- (1) $\pi \in \partial_e A(P(\mathcal{N}_1), P(\mathcal{N}_2))$.
- (2) $A(\pi) : C(\mathcal{N}_2) \rightarrow C(\mathcal{N}_1)$ is an algebra homomorphism.
- (3) $A(\pi) : C(\mathcal{N}_2) \rightarrow C(\mathcal{N}_1)$ is a lattice homomorphism.
- (4) $\pi(\mathcal{N}_1) \subset \mathcal{N}_2$.

We prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1) \Leftarrow (4) \Leftarrow (2)$.

Suppose π is extreme, then $A(\pi)$ is extreme in the set of unit-preserving positive linear operators from $C(\mathcal{N}_2)$ into $C(\mathcal{N}_1)$, and conversely. Suppose $a \in C(\mathcal{N}_2)$ and that $(1/4)1_{\mathcal{N}_2} \leq a \leq (3/4)1_{\mathcal{N}_2}$, then $(1/4)1_{\mathcal{N}_1} \leq A(\pi)a \leq (3/4)1_{\mathcal{N}_1}$.

If we define

$$T_1(b) = [A(\pi)a]^{-1}[A(\pi)(ab)], \quad T_2(b) = 4/3[A(\pi)b] - 1/3T_1(b),$$

then T_1 and T_2 are unit preserving and positive. As

$A(\pi) = 1/4T_1 + 3/4T_2$, $T_1 = T_2 = A(\pi)$. I.e. $[A(\pi)(ab)] = [A(\pi)(a)][A(\pi)(b)]$. If a is any element of $C(\mathcal{N}_2)$, there are positive integers m and n such that

$$(1/4)1_{\mathcal{N}_2} \leq (a - m1_{\mathcal{N}_2})/n \leq (3/4)1_{\mathcal{N}_2}.$$

Now $A(\pi)((a - m1_{\mathcal{N}_2})/n.b) = [A(\pi)((a - m1_{\mathcal{N}_2})/n)][A(\pi)(b)]$ so rearranging we have $A(\pi)(ab) = [A(\pi)(a)][A(\pi)(b)]$ as required.

If $A(\pi)$ is an algebra homomorphism, then for each $a \in C(\mathcal{N}_2)$,

$$[A(\pi)(a)]^2 = [A(\pi)(a^2)] = [A(\pi)(|a|^2)] = [A(\pi)(|a|)]^2.$$

As $A(\pi)$ is positive, $|A(\pi)(a)| = A(\pi)(|a|)$, so $A(\pi)$ is a lattice homomorphism.

Now suppose $A(\pi)$ is a lattice homomorphism, and that $A(\pi) = \lambda T_1 + (1 - \lambda)T_2$, where $0 < \lambda < 1$ and T_1, T_2 are positive and unit preserving. By Lemma V.2.1 T_1 and T_2 are lattice

homomorphisms. Thus for all $a \in C(\mathcal{N}_2)$,

$$\begin{aligned} |\lambda T_1(a) + (1 - \lambda)T_2(a)| &= (\lambda T_1 + (1 - \lambda)T_2)(|a|) \\ &= \lambda T_1(|a|) + (1 - \lambda)T_2(|a|) \\ &= \lambda |T_1(a)| + (1 - \lambda) |T_2(a)|. \end{aligned}$$

If $T_1(a) \neq T_2(a)$ for some $a \in C(\mathcal{N}_2)$ then, as $T_i^{-1}\mathcal{N}_2 = \mathcal{N}_1$, we can find $b \in C(\mathcal{N}_2)$ and $w \in \mathcal{N}_1$ such that $T_1(b)(w) > 0 > T_2(b)(w)$.

However we have:

$$\lambda |T_1(b)(w)| + (1 - \lambda) |T_2(b)(w)| = |\lambda T_1(b)(w) + (1 - \lambda)T_2(b)(w)|$$

which gives a contradiction. Thus $T_1 = T_2$, so $A(\pi)$ and hence π are extreme.

If (4) holds, and $\pi = \lambda \pi_1 + (1 - \lambda) \pi_2$ with $0 < \lambda < 1$ and $\pi_i \in A(P(\mathcal{N}_1), P(\mathcal{N}_2))$, then $\pi(w) = \lambda \pi_1(w) + (1 - \lambda) \pi_2(w)$ for all $w \in P(\mathcal{N}_1)$. In particular this holds if $w \in \mathcal{N}_1$, when $\pi(w) \in \mathcal{N}_2$, the set of extreme points of $P(\mathcal{N}_2)$. Hence $\pi_1(w) = \pi_2(w)$ if $w \in \mathcal{N}_1$, and by the Krein-Milman theorem $\pi_1 = \pi_2$, so π is extreme.

Finally, let $A(\pi)$ be an algebra homomorphism, and let $w \in \mathcal{N}_2$. If $a, b \in C(\mathcal{N}_2)$, then

$$\begin{aligned} (ab)(\pi w) &= [A(\pi)(ab)](w) \\ &= [(A(\pi)(a))(A(\pi)(b))](w) \\ &= [(A(\pi)(a))(w)][(A(\pi)(b))(w)] \\ &= a(\pi w)b(\pi w). \end{aligned}$$

Thus πw is a multiplicative linear functional on $C(\mathcal{N}_2)$, so $\pi w \in \mathcal{N}_2$ (by (2) $\Rightarrow$ (1) with $\mathcal{N}_1$ a singleton). I.e. $\pi(\mathcal{N}_1) \subset \mathcal{N}_2$ and the proof is complete.

In establishing the equivalence of (1) and (4) we can drop the assumption that the domain be a Bauer simplex, as long as we require also that the co-domain be metrisable. In fact we have:

THEOREM V.2.3. Let S be a compact simplex, $\mathcal{N}$ a compact metric space and $\pi \in A(S, P(\mathcal{N}))$. The following are equivalent:

$$(1) \quad \pi \in \partial_e A(S, P(\mathcal{N})).$$

(2) $f, g \in C(\mathcal{N}) \Rightarrow A(\pi)(f \vee g)$ is the supremum in $A(S)$ of $A(\pi)(f)$ and $A(\pi)(g)$.

$$(3) \quad \pi(\partial_e S) \subset \mathcal{N}.$$

(1) $\Rightarrow$ (3). Define $\Phi : P(\mathcal{N}) \rightarrow 2^{M(\mathcal{N})}$ by

$$\Phi(\mu) = \{ \nu \in M(\mathcal{N}) : 2\mu \geq \nu \geq 0 \}.$$

We claim that Φ is lower semi-continuous for the weak* topologies on $P(\mathcal{N})$ and $M(\mathcal{N})$. I.e. we have to show that if $\mu \in P(\mathcal{N})$, $\nu \in \Phi(\mu)$ and U is a neighbourhood of ν then there is a neighbourhood V of μ such that $\Phi(\mu') \cap U \neq \emptyset$ whenever $\mu' \in V$.

Let U be the neighbourhood of ν ,

$$\{ \nu' \in M(\mathcal{N}) : \left| \int_{\mathcal{N}} f_i d\nu' - \int_{\mathcal{N}} f_i d\nu \right| \leq 1, \\ f_i \in C(\mathcal{N}), 1 \leq i \leq n \}. \quad (+)$$

Suppose $\{\mu_\alpha\}$ is a net in $P(\mathcal{N})$ converging to μ for which $\Phi(\mu_\alpha) \cap U = \emptyset$ for every α , we obtain a contradiction. From the Radon-Nikodym theorem there is a Borel function g on $\mathcal{N}$ such that $0 \leq g \leq 2$ and $d\nu = g d\mu$. Choose $g_1 \in C(\mathcal{N})$ with $0 \leq g_1 \leq 2$ and which satisfies

$$\int_{\mathcal{N}} |g - g_1| d\mu \leq [2 \|f_i\|]^{-1} \quad (1 \leq i \leq n, \|f_i\| \neq 0). \quad (++)$$

(This exists as the unit ball of any Banach space E is $\sigma(E^{**}, E^*)$ -dense in that of E^{**}). If we define the measure

$\nu_\alpha \in M(\mathcal{N})$ by $d\nu_\alpha = g_1 d\mu_\alpha$ then $\nu_\alpha \in \Phi(\mu_\alpha)$, and

$$\lim_{\alpha} \int_{\mathcal{N}} f_i d\nu_\alpha = \lim_{\alpha} \int_{\mathcal{N}} f_i g_1 d\mu_\alpha \\ = \int_{\mathcal{N}} f_i g_1 d\mu. \quad (+++)$$

We have

$$\left| \int_{\mathcal{N}} f_i d\nu - \int_{\mathcal{N}} f_i d\nu_\alpha \right| \leq \left| \int_{\mathcal{N}} f_i g d\mu - \int_{\mathcal{N}} f_i g_1 d\mu \right| + \\ \left| \int_{\mathcal{N}} f_i g_1 d\mu - \int_{\mathcal{N}} f_i g_1 d\mu_\alpha \right|. \quad (++++)$$

From (+), (++) , (+++) and (++++) we deduce that ν_α is eventually in U and this is the desired contradiction.

Now let $\bar{\Psi} : P(\mathcal{N}) \longrightarrow 2^{P(\mathcal{N})}$ be defined by

$\bar{\Psi}(\mu) = \bar{\Phi}(\mu) \cap P(\mathcal{N})$. It is easy to see that $\bar{\Psi}$ is affine and that $\bar{\Psi}(\mu)$ is weak*-closed in $P(\mathcal{N})$ for each $\mu \in P(\mathcal{N})$. We show also that $\bar{\Psi}$ is lower semi-continuous. Take $\mu \in P(\mathcal{N})$, $\nu \in \bar{\Psi}(\mu)$ and $\{\mu_\alpha\}$ a net in $P(\mathcal{N})$ converging to μ . By the lower semi-continuity of $\bar{\Phi}$ there are measures $\nu_\alpha \in \bar{\Phi}(\mu_\alpha)$ such that the net $\{\nu_\alpha\}$ converges to ν . Define

$$\begin{aligned} \nu'_\alpha &= \nu_\alpha / \|\nu_\alpha\| & (\|\nu_\alpha\| \geq 1) \\ &= [2(1 - \|\nu_\alpha\|)\mu + \nu_\alpha] / (2 - \|\nu_\alpha\|) & (\|\nu_\alpha\| < 1). \end{aligned}$$

Clearly $\nu'_\alpha \geq 0$, $2\mu_\alpha - \nu'_\alpha \geq 0$ and $\|\nu'_\alpha\| = 1$, so that $\nu'_\alpha \in \bar{\Psi}(\mu_\alpha)$. Since $\lim_\alpha \|\nu_\alpha\| = \lim_\alpha \nu_\alpha(1_{\mathcal{N}}) = \nu(1_{\mathcal{N}}) = 1$, the net $\{\nu'_\alpha\}$ is weak* convergent to ν . This suffices to prove that $\bar{\Psi}$ is lower semi-continuous, since $\nu'_\alpha \in \bar{\Psi}(\mu)$.

The map $\bar{\Psi} \circ \pi$ is lower semi-continuous and affine. If $\pi(\partial_e S) \not\subset \mathcal{N}$, then for some $k \in \partial_e S$, $(\bar{\Psi} \circ \pi)(k) \neq \{\pi(k)\}$. We can now obtain a continuous affine selection π' of $\bar{\Psi} \circ \pi$ with $\pi'(k) \neq \pi(k)$, by application of II.1.8, since $\mathcal{N}$ is metrisable, and hence so is $P(\mathcal{N})$. It is now easy to see that $\pi', 2\pi - \pi' \in A(S, P(\mathcal{N}))$, $\pi' \neq \pi$ and this contradicts the extremality of π .

(3) $\implies$ (2). If $f, g \in C(\mathcal{N})$ and $k \in \partial_e S$ then it is obvious that $A(\pi)(f \vee g) \geq A(\pi)(f), A(\pi)(g)$. Suppose $h \in A(S)$ with $h \geq A(\pi)(f), A(\pi)(g)$. Then we have

$$\begin{aligned} [A(\pi)(f \vee g)](k) &= (f \vee g)(\pi k) = f(\pi k) \vee g(\pi k) \\ &= [A(\pi)(f)](k) \vee [A(\pi)(g)](k) \\ &\leq h(k). \end{aligned}$$

By the Krein-Milman theorem, $A(\pi)(f \vee g) \leq h$. I.e. $A(\pi)(f \vee g)$ is

the supremum of $A(\pi)(f)$ and $A(\pi)(g)$.

(2) $\implies$ (3). Suppose $f, g \in C(\mathcal{L})$ and $k \in \partial_e S$, then

$$\begin{aligned} (f \vee g)(\pi k) &= [(A(\pi)(f)) \vee (A(\pi)(g))](k) \\ &= [A(\pi)(f)](k) \vee [A(\pi)(g)](k) \\ &= f(\pi k) \vee g(\pi k). \end{aligned}$$

I.e. πk is a lattice homomorphism of $C(\mathcal{L})$ into $\mathbb{R}$, so $\pi k \in \mathcal{L}$ by Theorem V.2.2.

(3) $\implies$ (1) is proved in the same way as (4) $\implies$ (1) of Theorem V.2.2.

If we no longer require that the codomain be a Bauer simplex, it will clearly not be necessary that extreme maps be extreme point preserving. It is not clear what a necessary and sufficient condition might be. In the absence of such, those maps that are extreme point preserving seem worthy of further study. This we shall do in the next section.

3 Extreme point preserving morphisms.

Throughout this section we shall deal with continuous affine maps π between compact simplexes S_1 and S_2 such that $\pi(\partial_e S_1) \subset \partial_e S_2$. We first characterise such maps in terms of the order properties of $A(\pi)$.

Let E, F be ordered vector spaces and let $T : E \rightarrow F$ be a positive linear operator. T is an R-homomorphism if whenever $x_1, x_2 \in E, y \in F$ and $Tx_1, Tx_2 \leq y$ there is $x_3 \in E$ with $x_3 \gg x_1, x_2$ and $Tx_3 \leq y$.

It is easily verified that if E and F are lattices then the R-homomorphisms are precisely the lattice homomorphisms.

We have the following result:

THEOREM V.3.1. Let S_1, S_2 be compact simplexes and $\pi \in A(S_1, S_2)$, then the following are equivalent:

- (1) $\pi(\partial_e S_1) \subset \partial_e S_2$.
- (2) $A(\pi)$ is an R-homomorphism.

(2) $\Rightarrow$ (1). Suppose $k \in \partial_e S_1$ and that we can write $\pi k = \lambda x_1 + (1 - \lambda)x_2$ with $0 < \lambda < 1$ and $x_1 \neq x_2$. By Hahn-Banach separation there is $a \in A(S_2)$ with $a(x_1) > 0, a(\pi k) \leq 0$. Using Edwards' theorem II.1.3 we can find $h \in A(S_1)$ with $h \geq Ta, 0$ and $h(k) = 0$. By (2) there is $b \in A(S_2)$ with $b \geq a, 0$ and $h \geq Tb$. It follows that $b(\pi k) = \lambda b(x_1) + (1 - \lambda)b(x_2) \geq \lambda a(x_1) + (1 - \lambda)0 > 0$. But $0 = h(k) \geq (Tb)(k) = b(\pi k)$ so we have a contradiction.

(1) $\Rightarrow$ (2). Suppose first that $a, b \in A(S_2), h > A(\pi)(a), A(\pi)(b)$. We show first that $[A(\pi)(a) \vee A(\pi)(b)]^\wedge = \inf \{A(\pi)(c) : c \in A(S_2), c \geq a, b\}$. Since both functions are u.s.c. and affine it will suffice, by I.2.2, to show that they coincide on $\partial_e S_1$. If $k \in \partial_e S_1$ then

$$\begin{aligned}
 [A(\pi)(a) \vee A(\pi)(b)]^{\wedge}(k) &= [A(\pi)(a) \vee A(\pi)(b)](k) \\
 &= [A(\pi)(a)](k) \vee [A(\pi)(b)](k) \\
 &= a(\pi k) \vee b(\pi k) \\
 &= (a \vee b)(\pi k) \\
 &= (a \vee b)^{\wedge}(\pi k) \\
 &= \inf \{c(\pi k) : c \in A(S_2), c \geq a, b\} \\
 &= \inf \{[A(\pi)(c)](k) : c \geq a, b\}
 \end{aligned}$$

(we use here the fact that if f is continuous and convex, then f and $\hat{f}$ coincide on $\partial_e S$, I.3.5). It follows that at each point of S_1 there is $c_k \in A(S_2)$ with $a, b < c_k$ and $h(k) > [A(\pi)(c_k)](k)$. By continuity both strict inequalities persist in a neighbourhood U_k of k . Applying compactness we can find $k_1, \dots, k_n \in S_1$ such that $\bigcup_{i=1}^n U_{k_i} \supset S_1$. Now choose $\varepsilon > 0$ so that $a + \varepsilon, b + \varepsilon \leq c_{k_i} - \varepsilon$ ($i = 1, 2, \dots, n$). If $a + \varepsilon, b + \varepsilon \leq c \leq c_{k_i} - \varepsilon$ then $a, b < c < c_{k_i}$. It is now clear that $A(\pi)(c) < h$. It remains now to remove the assumption that $A(\pi)(a), A(\pi)(b) < h$.

If $A(\pi)(a), A(\pi)(b) \leq h$, we construct a sequence $(c_n) \subset A(S_2)$ such that

$$\begin{aligned}
 (a) \quad & \|c_n - c_{n-1}\| < 2^{-n} \quad (n = 1, 2, \dots) \\
 (b) \quad & (a - 2^{-n}), (b - 2^{-n}) < c_n \quad (n = 0, 1, \dots) \\
 (c) \quad & A(\pi)(c_n) < h + 2^{-n} \quad (n = 0, 1, \dots).
 \end{aligned}$$

By the earlier part of the proof, we can find $c_0 \in A(S_2)$ with $(a - 1), (b - 1) < c_0, A(\pi)(c_0) < h + 1$. Suppose $c_0, c_1, \dots, c_n$ have been defined, then we have

$$\begin{aligned}
 & A(\pi)(a - 2^{-(n+1)}), A(\pi)(b - 2^{-(n+1)}), A(\pi)(c_n - 2^{-(n+1)}) \\
 & < (h + 2^{-(n+1)}), A(\pi)(c_n + 2^{-(n+1)}).
 \end{aligned}$$

Now use the weak R.S.P. for $A(S_1)$ to obtain f_{n+1} satisfying

$$\begin{aligned}
 & A(\pi)(a - 2^{-(n+1)}), A(\pi)(b - 2^{-(n+1)}), A(\pi)(c_n - 2^{-(n+1)}) \\
 & < f_{n+1} < (h + 2^{-(n+1)}), A(\pi)(c_n + 2^{-(n+1)}).
 \end{aligned}$$

We know that there is c'_{n+1} with

$$a - 2^{-(n+1)}, b - 2^{-(n+1)}, c_n - 2^{-(n+1)} < c'_{n+1}$$

and

$$A(\pi)(c'_{n+1}) < f_{n+1}.$$

Now if we apply the weak R.S.P. for $A(S_2)$ we may find c_{n+1} with

$$a - 2^{-(n+1)}, b - 2^{-(n+1)}, c_n - 2^{-(n+1)} < c_{n+1} < c'_{n+1}, c_n + 2^{-(n+1)}.$$

With this choice of c_{n+1} conditions (a), (b) and (c) are certainly satisfied. As $A(S_2)$ is complete and (c_n) is Cauchy, the sequence has a limit $c \in A(S_2)$. It is clear that $a, b \leq c$ and that $A(\pi)(c) \leq h$, so that the proof is complete.

One reason that the extreme point preserving maps are easier to handle than the extreme maps is that we can compose two such maps to obtain a third of the same nature. It follows fairly easily that the compact simplexes with extreme point preserving continuous affine maps between them form a category. We can prove that projective simplexes satisfy (or define them by) Proposition V.1.6, with "compact convex set" replaced by "simplex", and with all morphisms required to be extreme point preserving.

THEOREM V.3.2. In the category of compact simplexes and extreme point preserving continuous affine maps, the projective elements are the finite dimensional simplexes.

If S is a finite dimensional simplex, X and Y are simplexes and $\psi : S \rightarrow X$, $\pi : Y \rightarrow X$ are continuous, affine and extreme point preserving, with π a surjection, let $k \in \partial_e S$. Then $\psi(k) \in \partial_e X$ and $\pi^{-1}(\psi(k))$ is a (non-empty) closed face of Y . By the Krein-Milman theorem this set has an extreme point y_k .

Choose such a y_k for each $k \in \partial_e S$, and define ψ to be the unique continuous affine map from S into Y with $\psi(k) = y_k$ for each $y \in \partial_e S$. Now $\psi = \pi \circ \varphi$ and φ has all the required properties.

Suppose now that S is projective. Let $k \in \partial_e S$ and define S_1 to be the convex hull of $\{(k, -1), (k, 1)\} \cup (S \times \{0\})$ in $E \times \mathbb{R}$ where S is embedded in the locally convex space E . It is clear that S_1 is a compact convex set, and that $\partial_e S_1 = \{(k, -1), (k, 1)\} \cup ((\partial_e S \setminus \{k\}) \times \{0\})$. We claim that S_1 is a simplex, to prove which it suffices, by I.1.13, to prove that $A(S_1)$ has the R.S.P. We may identify $A(S_1)$ with the subspace B of $\mathbb{R} \times \mathbb{R} \times A(S)$ consisting of all triples (x', x'', a) with $2a(k) = x' + x''$, with the order $(x', x'', a) \geq (0, 0, 0)$ if and only if $x', x'' \geq 0$ and $a \geq 0$. Suppose $(x'_1, x''_1, a_1) \geq (x'_2, x''_2, a_2) \geq (z'_1, z''_1, c_1), (z'_2, z''_2, c_2)$ then we may find $y', y'' \in \mathbb{R}$ with $x'_1, x'_2 \geq y' \geq z'_1, z'_2$ and $x''_1, x''_2 \geq y'' \geq z''_1, z''_2$. We may now apply Edwards' theorem to obtain $b \in A(S)$ with $a_1, a_2 \geq b \geq c_1, c_2$ and $b(k) = (1/2)(y' + y'')$. It is now clear that

$$(x'_1, x''_1, a_1), (x'_2, x''_2, a_2) \geq (y', y'', b) \geq (z'_1, z''_1, c_1), (z'_2, z''_2, c_2)$$

and the assertion is proved.

We show first that $\partial_e S$ is discrete. Having chosen $k \in \partial_e S$ and defined S_1 as above, let π denote the natural projection of S_1 onto S . As S is projective, and π is extreme point preserving and affine, there is a continuous affine extreme point preserving $\varphi : S \rightarrow S_1$ with $\pi \circ \varphi$ equal to the identity on S . Clearly if $x \in \partial_e S \setminus \{k\}$, then $\varphi(x) = (x, 0)$ as $(x, 0)$ is the only point mapped onto x by π . If (x_γ) is a net in $\partial_e S \setminus \{k\}$ converging to k , $\varphi(k) = \lim_\gamma \varphi(x_\gamma) = \lim_\gamma (x_\gamma, 0) = (k, 0)$, by the continuity of φ . As $(k, 0)$ is not extreme, we have a contradiction.

We prove next that $F = \{k\}$ is closed. If $p \in F$ there

is a Borel probability measure μ representing p and supported by $\overline{\partial_e S}$. If $\mu(\{k\}) \neq 0$ we could express p as a proper convex combination of k and another point, which is impossible as p belongs to a face not containing k . It follows that μ is supported by $\overline{\partial_e S \setminus \{k\}} = \overline{\partial_e S} \setminus \{k\}$ so that $p \in \overline{\text{co}(\partial_e S \setminus \{k\})}$.

As φ is continuous and affine, $\varphi(p) = (p, 0)$ for each $p \in F$. By continuity $\varphi(p) = (p, 0)$ whenever $p \in \overline{F}$. If $p \in \overline{F}$ we can write $p = \alpha q + (1 - \alpha)k$ with $q \in F$ and $0 \leq \alpha \leq 1$, by Proposition III.2.1.

Now we have

$$\begin{aligned} (p, 0) &= \varphi(p) = \varphi(\alpha q + (1 - \alpha)k) = \alpha \varphi(q) + (1 - \alpha) \varphi(k) \\ &= \alpha(q, 0) + (1 - \alpha)(k, \pm 1) = (\alpha q + (1 - \alpha)k, \pm(1 - \alpha)), \end{aligned}$$

using the fact that $\varphi(k) = (k, 1)$ or $(k, -1)$. Hence $\alpha = 1$, and $p \in F$. I.e. $F = \overline{F}$.

It follows that $\partial_e S$ is closed, for if $x \in \overline{\partial_e S} \setminus \partial_e S$, then $x \in \{k\}$ ' as this set is closed. Now $x \in \bigcap_{k \in \partial_e S} \{k\}$ ', and the latter set is a closed face of S . As it is non-empty this set must have an extreme point, k_0 . But $k_0 \notin \{k_0\}$ ' $\supset \bigcap_{k \in \partial_e S} \{k\}$ ' so we have the result. As we already know $\partial_e S$ is discrete, $\partial_e S$ is a finite set, so that S must be finite dimensional, as claimed.

CHAPTER VI

TENSOR PRODUCTS

1 Projective tensor product.

Let K_1, K_2 be compact convex sets and C a convex subset of a locally convex space, then $BA(K_1, K_2; C)$ denotes the set of all continuous function $\pi : K_1 \times K_2 \rightarrow C$ such that for each $k_1 \in K_1$ and each $k_2 \in K_2$ the functions $k \mapsto \pi(k_1, k)$ on K_2 and $k \mapsto \pi(k, k_2)$ on K_1 are affine. We write $BA(K_1, K_2)$ in place of $BA(K_1, K_2, \mathbb{R})$.

A projective tensor product of K_1 and K_2 is a pair (X, π) where X is a compact convex set and $\pi \in BA(K_1, K_2; X)$ such that for each compact convex set Y and each $\varphi \in BA(K_1, K_2; Y)$ there is a unique $\psi \in A(X, Y)$ with $\varphi = \psi \circ \pi$.

LEMMA VI.1.1. If (X, π) and (X', π') are projective tensor products of K_1 and K_2 then X and X' are affinely homeomorphic.

From the definition of projective tensor product there exists $\psi \in A(X, X')$ with $\pi' = \psi \circ \pi$. Similarly there is $\psi' \in A(X', X)$ with $\pi = \psi' \circ \pi'$. Hence $\pi = (\psi' \circ \psi) \circ \pi$ and $\pi' = (\psi \circ \psi') \circ \pi'$. Again using the definition with $X = Y$ and noting that $\pi = i_X \circ \pi$ (where i_X is the identity on X) we have $\psi' \circ \psi = i_X$ because of the uniqueness condition. Similarly $\psi \circ \psi' = i_{X'}$, so ψ and ψ' are the required affine homeomorphisms.

To show that such a projective tensor product always does exist, we proceed as follows. We note that $BA(K_1, K_2)$ is a closed subspace of $C(K_1 \times K_2)$ which contains the constants and separates the points of $K_1 \times K_2$. We may thus form its state space. If $(k_1, k_2) \in K_1 \times K_2$ then there is a unique

element of $S[BA(K_1, K_2)]$, that we denote by $w(k_1, k_2)$ such that $w(k_1, k_2)(a) = a(k_1, k_2)$ for each $a \in BA(K_1, K_2)$. It is easily verified that the function $(k_1, k_2) \mapsto w(k_1, k_2)$ is in $BA(K_1, K_2; S[BA(K_1, K_2)])$.

If $a \in A(K_1)$ and $b \in A(K_2)$ we define $a \otimes b \in BA(K_1, K_2)$ by $(a \otimes b)(k_1, k_2) = a(k_1)b(k_2)$ for $(k_1, k_2) \in K_1 \times K_2$. We let $A(K_1) \otimes A(K_2)$ denote the subspace $\left\{ \sum_{i=1}^n a_i \otimes b_i : a_i \in A(K_1), b_i \in A(K_2), n = 1, 2, \dots \right\}$ of $BA(K_1, K_2)$. The maps $T_1 : a \mapsto a \otimes 1$ and $T_2 : b \mapsto 1 \otimes b$ from $A(K_1)$ and $A(K_2)$ respectively into $BA(K_1, K_2)$ are order homomorphisms (in the sense of page 13). We denote by π_i the restriction of T_i^* to $S[BA(K_1, K_2)]$.

Before we show that this gives us the desired tensor product we look at the extremal structure of this set.

PROPOSITION VI.1.2. $w(\partial_e K_1 \times \partial_e K_2) = \partial_e(S[BA(K_1, K_2)])$.

Each extreme point of $S[BA(K_1, K_2)]$ is of the form $w(k_1, k_2)$ for some $k_i \in K_i$ by Corollary I.1.7. Clearly for $w(k_1, k_2)$ to be extreme we must have k_1 and k_2 extreme, since the functions in $BA(K_1, K_2)$ separate the points of $K_1 \times K_2$.

Now suppose $(k_1, k_2) \in \partial_e K_1 \times \partial_e K_2$, then $\pi_1^{-1}(k_1) \cap \pi_2^{-1}(k_2)$ is a closed face of $S[BA(K_1, K_2)]$. Clearly its only extreme point can be $w(k_1, k_2)$, so we have $\pi_1^{-1}(k_1) \cap \pi_2^{-1}(k_2) = \{w(k_1, k_2)\}$ by the Krein-Milman theorem. Thus $w(k_1, k_2)$ is an extreme point of $S[BA(K_1, K_2)]$.

THEOREM VI.1.3. The pair $(S[BA(K_1, K_2)], w)$ is a projective tensor product of K_1 and K_2 .

Suppose $\psi \in BA(K_1, K_2; Y)$ we define $U : A(Y) \rightarrow BA(K_1, K_2)$ by $(Ua)(k_1, k_2) = a(\psi(k_1, k_2))$. It is easily verified that U is an order homomorphism. Let ψ denote the restriction of

U^* to $S[BA(K_1, K_2)]$. If $(k_1, k_2) \in K_1 \times K_2$ then for $a \in A(Y)$ we have

$$\begin{aligned} [(\psi \circ \omega)(k_1, k_2)](a) &= \omega(k_1, k_2)(Ua) \\ &= (Ua)(k_1, k_2) \\ &= a(\varphi(k_1, k_2)) \end{aligned}$$

so that $\psi \circ \omega = \varphi$. If also $\psi' \in A(S[BA(K_1, K_2)], Y)$ with $\psi' \circ \omega = \varphi$ then clearly ψ and ψ' must coincide on $\omega(K_1 \times K_2)$. By Proposition VI.1.2 this means that ψ and ψ' coincide on the extreme points of $S[BA(K_1, K_2)]$, and hence coincide by the Krein-Milman theorem. The proof is thus complete.

We shall henceforth denote $S[BA(K_1, K_2)]$ by $K_1 \otimes K_2$.

It is obviously desirable to know something more about the facial structure of $K_1 \otimes K_2$. Proposition VI.1.2 tells us the extreme points of this set. More generally we have:

PROPOSITION VI.1.4. Let K_1 and K_2 be compact convex sets, and let F_i be a closed face of K_i for $i = 1, 2$. If $F = \pi_1^{-1}(F_1) \cap \pi_2^{-1}(F_2)$ then F is a closed face of $K_1 \otimes K_2$ and $F = \overline{\text{co}}(\omega(F_1 \times F_2))$.

It is clear that F is a closed face. By the Krein-Milman theorem F is the closed convex hull of its set of extreme points. The extreme points of F are the points $\omega(k_1, k_2)$ for $k_i \in \partial_e K_i$ which lie in F . Now $\pi_i(\omega(k_1, k_2)) = k_i$, so $\partial_e F = \{\omega(k_1, k_2) : k_i \in \partial_e F_i\}$. Since it is clear that $\omega(F_1 \times F_2) \subset F$ we have $F = \overline{\text{co}}(\omega(F_1 \times F_2))$.

PROPOSITION VI.1.5. With the notation of Proposition VI.1.4 if F is a split face of $K_1 \otimes K_2$ then F_1 and F_2 are split faces.

Suppose $a \in A(K_1)$ and $a|_{F_1} \geq 0$. By III.2.13 it will suffice to prove that $(a\chi_{F_1})^\wedge$ is affine on K_1 . We know

that $((a \otimes 1)\chi_F)$ is u.s.c. and affine on $K_1 \otimes K_2$ since $a \otimes 1$ is non-negative on $\mathcal{W}(F_1 \times F_2)$ and hence on F , and by applying III.2.13 for the split face F . Now fixing $k_2 \in \partial_e F_2$, the function $g(k_2) : k \mapsto ((a \otimes 1)\chi_F)^\wedge(\mathcal{W}(k_1, k_2))$ is u.s.c. and affine on K_1 . On F_1 $g(k_2)$ agrees with a , and since $\mathcal{W}(\partial_e F_1' \times \partial_e F_2) \subset F'$ we also have $g(k_2) = 0$ on $\partial_e F_1'$.

Since $g(k_2)$ and $(a\chi_{F_1})^\wedge$ agree on $\partial_e K_1$ it follows from the facts that $g(k_2)$ is u.s.c. and affine and that $(a\chi_{F_1})^\wedge$ is u.s.c. and concave, together with Proposition I.2.2, that $g(k_2) \leq (a\chi_{F_1})^\wedge$. But we certainly have $g(k_2) \geq (a\chi_{F_1})^\wedge$ so $(a\chi_{F_1})^\wedge$ is affine. This completes the proof for F_1 . That for F_2 is similar.

An obvious question of interest is to determine when $K_1 \otimes K_2$ is a simplex. We shall return to this question later.

2 Biprojective tensor product.

We have an alternative product to consider. Although this product is of rather less use than the projective tensor product it does have the advantage that we can find out rather more about it. Later we shall see that on many occasions the two products coincide.

We denote the closure of $A(K_1) \otimes A(K_2)$ in $BA(K_1, K_2)$ by $A(K_1) \hat{\otimes} A(K_2)$. The state space of this, $K_1 \Delta K_2$, is the biprojective tensor product of K_1 and K_2 .

We again let T_i be the natural injections of $A(K_i)$ into $A(K_1) \hat{\otimes} A(K_2)$, and now $\hat{\pi}_i$ denotes the restriction of T_i^* to $K_1 \Delta K_2$. The restriction map $P : K_1 \otimes K_2 \rightarrow K_1 \Delta K_2$ is continuous, affine and onto by the Hahn-Banach theorem. If $(k_1, k_2) \in K_1 \times K_2$ we denote by $k_1 \otimes k_2$ the element $P(\omega(k_1, k_2))$ of $K_1 \Delta K_2$. We may again describe the extremal structure of $K_1 \Delta K_2$ in simple terms.

PROPOSITION VI.2.1. $\partial_e(K_1 \Delta K_2) = \{k_1 \otimes k_2 : k_i \in \partial_e K_i\}$.

That all the extreme points of $K_1 \Delta K_2$ are of the form $k_1 \otimes k_2$ for $k_i \in \partial_e K_i$ is proved as in Proposition VI.1.2. If $x \in \partial_e(K_1 \Delta K_2)$ then $P^{-1}(x)$ is a closed face of $K_1 \otimes K_2$. All the extreme points of this are of the form $\omega(k_1, k_2)$ for $k_i \in K_i$. Since the functions in $A(K_1) \hat{\otimes} A(K_2)$ separate the points of $K_1 \times K_2$, $P^{-1}(x)$ must be a singleton. Hence $x = k_1 \otimes k_2$ for $k_i \in \partial_e K_i$.

More generally we have:

PROPOSITION VI.2.2. Let F_1, F_2 be closed faces of K_1, K_2 respectively, and let $F = \hat{\pi}_1^{-1}(F_1) \cap \hat{\pi}_2^{-1}(F_2)$. Then F is a closed face of $K_1 \Delta K_2$ and $F = \overline{\text{co}} \{k_1 \otimes k_2 : k_i \in F_i\}$. Furthermore F is affinely homeomorphic to $F_1 \Delta F_2$.

The proof of the first part of the result is identical to that of Proposition VI.1.4. Define $\psi : F_1 \Delta F_2 \longrightarrow K_1 \Delta K_2$ by

$$(\psi x)(a) = x(a|_{F_1 \times F_2}) \quad (x \in F_1 \Delta F_2, a \in A(K_1) \hat{\otimes} A(K_2)).$$

We denote the natural embedding of $F_1 \times F_2$ into $F_1 \Delta F_2$ by $(k_1, k_2) \longmapsto k_1 \otimes k_2$. Certainly ψ is continuous and affine, so that $\psi(F_1 \Delta F_2)$ is a closed convex subset of $K_1 \Delta K_2$. If $x \in \partial_e(F_1 \Delta F_2)$ then $x = k_1 \otimes k_2$ with $k_i \in \partial_e F_i$, so that then:

$$(\psi x)(a) = a(k_1, k_2) = (k_1 \otimes k_2)(a) \quad (a \in A(K_1) \hat{\otimes} A(K_2)).$$

Thus $\psi x \in F$. It follows by the Krein-Milman theorem that $\psi(F_1 \Delta F_2) \subset F$.

Conversely if $y \in \partial_e F$ then $y = k_1 \otimes k_2$ for $k_i \in \partial_e F_i$ as F is a face of $K_1 \otimes K_2$. Since $k_1 \otimes k_2 \in \partial_e(F_1 \Delta F_2)$ we have that $y \in \psi(F_1 \Delta F_2)$. Again by the Krein-Milman theorem we must in fact have $\psi(F_1 \Delta F_2) = F$.

Finally that ψ is (1,1) is because $A(K_i)|_{F_i}$ is dense in $A(F_i)$, so that $A(K_1) \hat{\otimes} A(K_2)|_{F_1 \times F_2}$ is dense in $A(F_1) \hat{\otimes} A(F_2)$. Thus ψ is the desired affine homeomorphism.

Before turning to split faces we need:

LEMMA VI.2.3. Let K_1, K_2 be compact convex sets, $a \in A_s(K_1)$ and $b \in A_s(K_2)$. Then there is a unique $c \in A_s(K_1 \Delta K_2)$, which we denote by $a \otimes b$, such that

$$c(k_1 \otimes k_2) = a(k_1)b(k_2) \quad ((k_1, k_2) \in K_1 \times K_2).$$

Suppose first that a and b are bounded, non-negative, u.s.c. and affine. Let $\{a_\alpha\} \subset A(K_1)^+$ and $\{b_\beta\} \subset A(K_2)^+$ be nets decreasing to a, b respectively pointwise. Then $\{a_\alpha \otimes b_\beta\}$ is a decreasing net in $(A(K_1) \hat{\otimes} A(K_2))^+$, and is bounded below since $\{a_\alpha\}$ and $\{b_\beta\}$ are. It follows that

the function c on $K_1 \Delta K_2$ defined by

$$c(x) = \inf_{\alpha, \beta} x(a_\alpha \otimes b_\beta) \quad (x \in K_1 \Delta K_2)$$

is u.s.c., affine and bounded. In particular if $k_1 \otimes k_2 \in K_1 \Delta K_2$ we have

$$c(k_1 \otimes k_2) = \inf_{\alpha, \beta} a_\alpha(k_1) b_\beta(k_2) = a(k_1) b(k_2).$$

If $a = a_1 - a_2$, $b = b_1 - b_2$ where a_i and b_i are bounded, u.s.c., non-negative and affine, then we may form $a_i \otimes b_j$ ($i, j = 1, 2$). Now defining $a \otimes b = a_1 \otimes b_1 + a_2 \otimes b_2 - (a_2 \otimes b_1 + a_1 \otimes b_2)$ we have a function with the desired properties.

In general, let $a \in A_S(K_1)$ and $b \in A_S(K_2)$. We may find a_n and b_n of the form just dealt with such that $\|b - b_n\| < 1/n$ and $\|a - a_n\| < 1/n$. Also we may find $c_n \in A_S(K_1 \Delta K_2)$ such that

$$c_n(k_1 \otimes k_2) = a_n(k_1) b_n(k_2) \quad (k_i \in K_i).$$

Now if $(k_1, k_2) \in \partial_e K_1 \times \partial_e K_2$ we have

$$|a(k_1) b(k_2) - c_n(k_1 \otimes k_2)| < 1/n^2 + 1/n(\|a\| + \|b\|).$$

It follows from Proposition I.3.2 that (c_n) is Cauchy. If c is the limit of (c_n) in $A_S(K_1 \Delta K_2)$, this obviously satisfies the requirements.

Uniqueness follows from I.3.2 as c is prescribed on $\partial_e(K_1 \Delta K_2)$.

THEOREM VI.2.4. With the notation of Proposition VI.2.2, F is a split face of $K_1 \Delta K_2$ if and only if F_1 and F_2 are split.

The proof that if F is a split face then F_1 and F_2 are is the same as that for Proposition VI.1.5. Suppose now that F_1 and F_2 are split faces. It will, by III.2.13, suffice to prove that if $a \in A(F)^+$ then a has an extension $\tilde{a} \in A_S(K_1 \Delta K_2)$ such that $a = 0$ on F' .

If $\epsilon > 0$ we can find $a_1, \dots, a_n \in A(F_1)$ and $b_1, \dots, b_n \in A(F_2)$ such that

$$\|a - \sum_{i=1}^n a_i \otimes b_i\| < \xi$$

(identifying F with $F_1 \Delta F_2$). By III.2.13 there exist $\tilde{a}_i \in A_S(K_1)$ and $\tilde{b}_i \in A_S(K_2)$ with $\tilde{a}_i = a_i$ on F_1 , $\tilde{a}_i = 0$ on F'_1 , $\tilde{b}_i = b_i$ on F_2 and $\tilde{b}_i = 0$ on F'_2 . By Lemma VI.2.3 we can form $\sum_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i$ in $A_S(K_1 \Delta K_2)$ which coincides with $\sum_{i=1}^n a_i \otimes b_i$ on $\{k_1 \otimes k_2 : k_i \in F_i\}$, while $\sum_{i=1}^n \tilde{a}_i \otimes \tilde{b}_i = 0$ on $\partial_e(K_1 \Delta K_2) \setminus \partial_e F$.

As $A_S(K_1 \Delta K_2)$ is complete and the functions in this space retain their norm on $\partial_e(K_1 \Delta K_2)$, this argument leads to the existence of $\tilde{a} \in A_S(K_1 \Delta K_2)$ with $\tilde{a} = a$ on $\{k_1 \otimes k_2 : k_i \in F_i\}$ and $\tilde{a} = 0$ on $\partial_e F' = \partial_e(K_1 \Delta K_2) \setminus \partial_e F$. It remains to show that $\tilde{a} = a$ on F and $\tilde{a} = 0$ on F' .

By I.3.2 as $\tilde{a}|_F$ and $a|_F$ coincide on $\partial_e F$ they coincide on F . To show that $\tilde{a} = 0$ on F' , let $b \in A(K_1 \Delta K_2)$ with $b > 0$ on $K_1 \Delta K_2$ and $b > a$ on F . Then $b \geq \tilde{a}$ on $\partial_e(K_1 \Delta K_2)$ so $b \geq \tilde{a}$ on $K_1 \Delta K_2$ by I.3.2. If $x \in K_1 \Delta K_2$ we have

$$\begin{aligned} (a \chi_F)^\wedge(x) &= \inf \{ b(x) : b \in A(K_1 \Delta K_2), b > a \chi_F \} \\ &\geq \tilde{a}(x) \geq 0. \end{aligned}$$

Since $(a \chi_F)^\wedge = 0$ on F' , we get $\tilde{a} = 0$ on F' , and the proof is complete.

We can now turn to the question of when $K_1 \Delta K_2$ is a simplex.

THEOREM VI.2.5. Let K_1 and K_2 be compact convex sets, then $K_1 \Delta K_2$ is a simplex if and only if both K_1 and K_2 are simplexes.

Suppose $K_1 \Delta K_2$ is a simplex. Choose $k_2 \in \partial_e K_2$ and let $F = \{k_1 \otimes k_2 : k_1 \in K_1\}$. Clearly F is affinely homeomorphic to K_1 . By Proposition VI.2.2 $\overline{\text{co}}(F) = F$ is a face of $K_1 \Delta K_2$. Hence F is a simplex by Corollary I.1.15, so K_1 also is a simplex. Similarly K_2 is also a simplex.

Now suppose K_1 and K_2 are simplexes. By Theorem II.1.2 it will suffice to show that there is an order homomorphism $E : C(K_1 \Delta K_2) \rightarrow A_s(K_1 \Delta K_2)$ that is the identity on $A(K_1 \Delta K_2)$. By the same result we may assume the same result we may assume the existence of similar operators E_1 and E_2 for K_1 and K_2 respectively.

Suppose $f \in C(K_1 \Delta K_2)$, define $\tilde{f} \in C(K_1 \times K_2)$ by $\tilde{f}(k_1, k_2) = f(k_1 \otimes k_2)$. If $a \in C(K_1)$, $b \in C(K_2)$ then $a \otimes b : (k_1, k_2) \mapsto a(k_1)b(k_2)$ is in $C(K_1 \times K_2)$. We denote by $C(K_1) \otimes C(K_2)$ the subspace of $C(K_1 \times K_2)$ generated by functions of this form. $1_{K_1} \otimes 1_{K_2} = 1_{K_1 \times K_2}$ and this subspace is clearly an algebra, so because the functions of $C(K_1) \otimes C(K_2)$ certainly separate the points of $K_1 \times K_2$, we may conclude that $C(K_1) \otimes C(K_2)$ is dense in $C(K_1 \times K_2)$.

We define $\tilde{E}$ on $C(K_1) \otimes C(K_2)$ by $\tilde{E}(\sum_{i=1}^n a_i \otimes b_i) = \sum_{i=1}^n (E_1 a_i) \otimes (E_2 b_i)$, which we may do by VI.2.3. It is easily checked that $\tilde{E}$ is well-defined. By the density of $C(K_1) \otimes C(K_2)$, the fact that $\tilde{E}(1_{K_1} \otimes 1_{K_2}) = 1_{K_1 \Delta K_2}$, that E is positive (by I.3.2 and the fact that E_i is positive) and the completeness of $A_s(K_1 \Delta K_2)$, we may extend E uniquely to the whole of $C(K_1 \times K_2)$ so that E is an order homomorphism. We now define $E : C(K_1 \Delta K_2) \rightarrow A_s(K_1 \Delta K_2)$ by $Ef = \tilde{E}\tilde{f}$.

It remains to show that if $f \in A(K_1 \Delta K_2)$ then $Ef = f$. By the definition of $K_1 \Delta K_2$, f is a limit of terms of $A(K_1) \otimes A(K_2)$. As E_i has such a property it follows that E is the identity on a dense subspace of $A(K_1 \Delta K_2)$. By continuity this persists on the whole of $A(K_1 \Delta K_2)$, and the proof is complete.

COROLLARY VI.2.6. Let K_1, K_2 be compact convex sets, then $K_1 \Delta K_2$ is a Bauer simplex if and only if both K_1 and K_2 are Bauer simplexes.

$\partial_e(K_1 \Delta K_2)$ is homeomorphic to the product $\partial_e K_1 \times \partial_e K_2$,
so is compact if and only if both factors $\partial_e K_i$ are compact.
The result is now immediate.

3 Applications.

In order to apply the theory that we have developed, we shall need to know when our two products coincide.

THEOREM VI.3.1. Let K_1, K_2 be compact convex sets, and suppose that $A(K_1)$ has the approximation property. Then $K_1 \otimes K_2$ is affinely homeomorphic to $K_1 \Delta K_2$.

We wish to know when $P : K_1 \otimes K_2 \longrightarrow K_1 \Delta K_2$ is $(1,1)$. Equivalently when is $A(K_1) \otimes A(K_2)$ dense in $BA(K_1, K_2)$? Suppose $A(K_1)$ has the approximation property. If $f \in BA(K_1, K_2)$ define $F \in A(K_2, A(K_1))$ by $(Fk_2)(k_1) = f(k_1, k_2)$ for $k_1 \in K_1$. It is easily verified that $f \longmapsto F$ is a linear isometry between $BA(K_1, K_2)$ and $A(K_2, A(K_1))$. Under this map the image of $A(K_1) \otimes A(K_2)$ is the set of functions in $A(K_2, A(K_1))$ with finite dimensional range. Given f , let $C = F(K_2)$, a compact convex subset of $A(K_1)$. So for each $\varepsilon > 0$ there is a continuous linear operator T on $A(K_1)$ with finite dimensional range and $\|Ta - a\| < \varepsilon$ for all $a \in C$. Then $T \circ F \in A(K_2, A(K_1))$ has finite dimensional range, and $\|T \circ F - F\| < \varepsilon$. It follows that $A(K_1) \otimes A(K_2)$ is dense in $BA(K_1, K_2)$, as claimed.

COROLLARY VI.3.2. If K is a compact convex set and S a simplex, then $S \otimes K$ is affinely homeomorphic to $S \Delta K$.

By Theorem II.4.8 $A(S)$ has the approximation property, so the result is immediate.

COROLLARY VI.3.3. If K_1, K_2 are compact convex sets, $K_1 \otimes K_2$ is a simplex if and only if both K_1 and K_2 are simplexes.

If $K_1 \otimes K_2$ is a simplex, a proof identical to that in Theorem VI.2.5 shows that K_1 and K_2 are simplexes. If K_1 and K_2 are simplexes then by Corollary VI.3.2 $K_1 \otimes K_2$ is

affinely homeomorphic to $K_1 \Delta K_2$ which is a simplex by Theorem VI.2.5.

We have a partial converse to Theorem VI.3.1.

THEOREM VI.3.4. If K_1 is a compact convex set, then $A(K_1)$ has the approximation property if and only if the restriction map $P : K_1 \otimes K_2 \longrightarrow K_1 \Delta K_2$ is (1,1) for each compact convex set K_2 .

We have already proved one implication. Suppose now that the restriction map is (1,1) for every K_2 . I.e. that $A(K_1) \otimes A(K_2)$ is dense in $BA(K_1, K_2)$ for every K_2 . Let C be a compact convex subset of $A(K_1)$. Define $g \in BA(K_1, C)$ by $g(k, c) = c(k)$ for $c \in C, k \in K_1$. By hypothesis there is $f \in A(K_1) \otimes A(C)$ such that $\|f - g\| < \varepsilon$. If φ and ψ denote the elements of $A(C, A(K_1))$ corresponding to f and g respectively, then ψ is the inclusion map of C into $A(K_1)$ and $\varphi(C)$ is finite dimensional. Let M be a finite dimensional subspace of $A(K_1)$ containing $\varphi(C)$. Then for each $c \in C, \|\varphi(c) - c\| = \|\varphi(c) - \psi(c)\| < \varepsilon$.

Now let $\{e_1, \dots, e_n\}$ be a basis for M . Then there are $\varphi_1, \dots, \varphi_n \in A(C)$ with $\varphi(c) = \sum_{i=1}^n \varphi_i(c) e_i$ for $c \in C$. By Corollary I.1.12 there are $\sigma_i \in A(K_1)^*$ and $\lambda_i \in \mathbb{R}$ with $\|\sigma_i + \lambda_i - \varphi_i\| \leq \varepsilon/n\|e_i\|$. Let $\sigma(a) = \sum_{i=1}^n \sigma_i(a) e_i$ for $a \in A(K_1)$. If $c \in C$ then

$$\begin{aligned} \|\sigma(c) - c\| &\leq \|(\sigma(c) + \sum_{i=1}^n \lambda_i e_i) - \sigma(c)\| + \\ &\quad \|\varphi(c) - (\sigma(c) + \sum_{i=1}^n \lambda_i e_i)\| + \|\varphi(c) - c\| \\ &\leq \|\sum_{i=1}^n \lambda_i e_i\| + \sum_{i=1}^n |\sigma_i(c) + \lambda_i - \varphi_i(c)| \|e_i\| \\ &\quad + \|\varphi(c) - c\|. \end{aligned}$$

But

$$\|\sum_{i=1}^n \lambda_i e_i\| = \|(\sigma(d) + \sum_{i=1}^n \lambda_i e_i) - \sigma(d)\|$$

$$\begin{aligned}
 &\leq \|(\sigma(d) + \sum_{i=1}^n \lambda_i e_i) - \varphi(d)\| + \|\varphi(d) - \sigma(d)\| \\
 &\leq \sum_{i=1}^n |(\lambda_i + \sigma_i(d)) - \varphi_i(d)| \|e_i\| + \|\varphi(d) - \sigma(d)\| \\
 &\leq 2\xi,
 \end{aligned}$$

choosing any $d \in C$. Hence

$$\|\sigma(c) - c\| \leq 4\xi \text{ for all } c \in C.$$

As σ is a linear operator with finite dimensional range, this completes the proof.

We are now in a position to apply our theory to obtain an extension of Theorem II.5.8.

THEOREM VI.3.5. Let F be a closed split face of a compact convex set K , and let E be a real Banach space. Let $p : K \rightarrow \mathbb{R}$ be concave, l.s.c. and strictly positive. Suppose $h \in A(F, E)$ with $\|h(k)\| \leq p(k)$ for each $k \in F$. If either E or $A(F)$ has the approximation property then there is $\tilde{h} \in A(K, E)$ such that $\tilde{h}|_F = h$ and $\|h(k)\| \leq p(k)$ for all $k \in K$.

Let S be the unit ball of E^* with the weak* topology. Norm $E \times \mathbb{R}$ by $\|(x, r)\| = \|x\| + |r|$. The map assigning to each pair the function that takes $s \in S$ to $s(x) + r$ is an isometry from $E \times \mathbb{R}$ onto $A(S)$. It is clear that if E has the approximation property, then so has $A(S)$.

We define $a \in BA(F, S)$ by $a(k, s) = s(h(k))$ for $k \in F$, $s \in S$. By Theorem VI.3.1 and the assumption that either E or $A(F)$ has the approximation property, a has an extension $a_1 \in A(F \Delta S)$. By VI.2.2 $\varphi : F \Delta S \rightarrow K \Delta S$ defined by

$$\varphi(x)(b) = x(b|_{F \times S}) \quad (x \in K \Delta S, b \in BA(K, S))$$

is an affine homeomorphism of $F \Delta S$ onto the subset $G =$

$\overline{\text{co}}\{k \otimes s : k \in F, s \in S\}$ of $K \Delta S$. Let $a_2 \in A(G)$ be the function

defined by

$$a_2(\psi x) = a_1(x) \quad (x \in K \Delta S).$$

The function $x \mapsto p(\hat{\pi}_1 x)$ is a concave l.s.c. strictly positive function on $K \Delta S$. If $x \in \partial_e G = \{k \otimes s : k \in \partial_e F, s \in \partial_e S\}$ we have

$$|a_2(x)| = |s(h(k))| \leq \|h(k)\| \leq p(k) = p(\hat{\pi}_1 x),$$

where $x = k \otimes s$. Since $x \mapsto |a_2(x)|$ is convex and continuous and $x \mapsto p(\hat{\pi}_1 x)$ is concave and l.s.c. it follows from Proposition I.2.2 that $|a_2| \leq p \circ \hat{\pi}_1$ on G .

By Theorem VI.2.4 G is a split face of $K \Delta S$, so there is $\tilde{a} \in A(K \Delta S)$ with $\tilde{a}|_G = a$ and $|\tilde{a}| \leq p \circ \hat{\pi}_1$ by Theorem III.2.15.

Define $c \in A(K, A(S))$ by

$$[c(k)](s) = \tilde{a}(k \otimes s) \quad (k \in K, s \in S).$$

Then for $k \in K$,

$$\begin{aligned} \|c(k)\| &= \sup_{s \in S} \|\tilde{a}(k \otimes s)\| \\ &\leq \sup_{s \in S} p(\hat{\pi}_1(k \otimes s)) \\ &= p(k). \end{aligned}$$

Now let R denote the isometry of $A(S)$ onto $E \times \mathbb{R}$, and Q the canonical projection from $E \times \mathbb{R}$ onto E . We now obtain the desired $h (= Q \circ R \circ c) \in A(K, E)$ such that

$$\|h(k)\| = \|(Q \circ R \circ c)(k)\| \leq \|c(k)\| \leq p(k)$$

for all $k \in K$. Finally to see that $\tilde{h}|_F = h$, suppose $k \in F$, $s \in S$ then

$$\begin{aligned} s(h(k)) &= s((Q \circ R \circ c)(k)) = c(k)(s) = \tilde{a}(k \otimes s) \\ &= a_2(k \otimes s) = s(h(k)). \end{aligned}$$

COROLLARY VI.3.6. Let F be a closed split face of a compact convex set K , and let E be a real Banach space. Suppose $h \in A(F, E)$ and either E or $A(F)$ has the approximation property, then h has an extension $\tilde{h} \in A(K, E)$ with $\|\tilde{h}\| = \|h\|$.

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