

LINEAR OPERATORS BETWEEN PARTIALLY ORDERED BANACH SPACES  
AND SOME RELATED TOPICS

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## A B S T R A C T

If  $X$  is a partially pre-ordered Banach space, with closed positive wedge, there are various questions of interest about this wedge. Is it generating? Is it normal? What ordering does it induce on  $X$ ? It is possible to deduce some properties of wedges from other properties of related wedges.

We consider Banach spaces of linear operators between two such spaces, and define  $T \geq 0$  iff  $Tx \geq 0$  whenever  $x \geq 0$ . We investigate properties of the positive wedge of these spaces. Firstly we look at spaces of all bounded linear operators. If the range is the reals, then a great deal is known, and a summary of these results, together with the definitions is given in Chapter I.

In Chapter II we look at the general case. We find conditions for the wedge to be normal, and certain cases in which it is generating. We also investigate when the wedge is one of some special forms that are of interest, and finally look at the order-structure of the space. The opportunity is taken to state some results for projective tensor products. Chapter III deals with the same problems for spaces of compact operators, generally with range a simplex space.

We deal with certain categories of partially ordered Banach spaces in Chapter IV. We determine injective elements in these categories, and also projective elements in certain categories of compact convex sets.

Finally we look at the structure of the spaces  $A(K)$  in Chapter V, obtaining a characterisation of the set of facially continuous functions as the union of all the subalgebras of  $A(K)$ . We also generalise the Banach-Stone theorem, and a result on direct sum decompositions of these spaces.

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## Chapter I

Partially ordered Banach spaces and their duality

In this chapter we present a brief summary of the basic results of the theory of partially ordered Banach spaces. In the first section we make the basic definitions in the theory of partially ordered vector spaces, and then study some partially ordered Banach spaces that arise naturally from these definitions. The second section will deal with the duality of the normality and generating properties for arbitrary partially ordered Banach spaces, and the third section with order properties of some spaces. This deals with both the duality of these properties, and also with the order properties of some of the special spaces considered.

1.1. Definitions and some special spaces.

Let  $V$  be a real vector space. A wedge  $W$  in  $V$  is a subset such that (1)  $W+W=W$ , and (2)  $\lambda W \subset W$  for all  $\lambda \in R_+$  (the non-negative reals). If  $W$  also satisfies (3)  $W \cap (-W) = \{0\}$  then it is termed a cone. The wedge  $W$  generates  $V$  if  $V = W - W$ .

A partial order on a vector space  $V$  is a relation " $\succcurlyeq$ " between pairs of elements of  $V$  satisfying (1')  $v_1 \succcurlyeq 0$  and  $v_2 \succcurlyeq 0 \Rightarrow v_1 + v_2 \succcurlyeq 0$ , (2')  $v \succcurlyeq 0$  and  $\lambda \in R_+ \Rightarrow \lambda v \succcurlyeq 0$ , and (3')  $v \succcurlyeq 0$  and  $0 \succcurlyeq v \Rightarrow v = 0$ . If the relation " $\succcurlyeq$ " satisfies only (1') and (2') then it is termed a partial preordering of  $V$ . We write " $v \preccurlyeq 0$ " for " $0 \succcurlyeq v$ ", and in this case term  $v$  negative. If  $v \succcurlyeq 0$  then we call  $v$  positive. Also we write  $u \succcurlyeq v$  to mean  $u - v \succcurlyeq 0$ .

The concepts of a partial preordering and of a wedge are intimately related. If  $W$  is a wedge then defining  $v \geq 0$  if and only if  $v \in W$  defines a preorder on  $V$ . Conversely if " $\geq$ " is a preorder on  $V$  then the set  $\{v : v \geq 0\}$  is a wedge; and " $\geq$ " arises from this wedge in the manner just described. " $\geq$ " is an ordering if and only if  $\{v : v \geq 0\}$  is a cone. The wedge  $\{v : v \geq 0\}$  is generating if and only if for each  $u \in V$ , there is a  $v \in V$  satisfying  $v \geq u, 0$ . We use the term "partially ordered vector space" interchangeably for a vector space with a distinguished cone, or with a partial order defined on it. We shall write  $V_+$  for the cone of positive elements of  $V$ .

$V$  is almost Archimedean if  $-\lambda v \leq u \leq \lambda v$  for some  $v \geq 0$  and all  $\lambda > 0$  implies  $u = 0$ . If  $u \leq 0$  whenever  $\lambda u \leq v$  for some  $v$  and all  $\lambda > 0$ , then  $V$  is termed Archimedean. If  $u \leq w$ , and  $u, w \in V$ , then  $[u, w]$  will denote the set  $\{v \in V : u \leq v \leq w\}$ . Such a set will be termed an order-interval. A subset  $A$  of  $V$  is order-bounded if it is contained in some order-interval, i.e. if there are  $u, w \in V$  such that  $u \leq a \leq w$  for all  $a \in A$ .

If  $U$  and  $V$  are partially ordered vector spaces, a linear operator  $T$  mapping  $U$  into  $V$  is bipositive if  $Tu \geq 0$  if and only if  $u \geq 0$ . If  $T$  is  $(1,1)$  and onto then  $T$  is an order-isomorphism, in which case  $U$  and  $V$  are order-isomorphic. If  $U$  and  $V$  are also normed spaces, and  $\|Tu\| = \|u\|$  for all  $u \in U$ , then  $U$  and  $V$  are isometrically order-isomorphic.

Let  $V$  be a partially ordered vector space. If  $u, v \in V$ , then  $w \in V$  is termed the supremum (in  $V$ ) of  $u$  and  $v$  if:

$$(a) w \geq u, v$$

$$(b) x \in V, x \geq u, v \Rightarrow x \geq w.$$

Such a  $w$  will be denoted by  $u \vee v$ . The infimum of  $u$  and  $v$  is

defined similarly to be the greatest lower bound for  $u$  and  $v$ , and denoted by  $u \wedge v$ . The positive part of  $u$ ,  $u^+$ , is defined to be  $u \vee 0$ , and the negative part of  $u$ ,  $u^-$ , is  $(-u) \vee 0$ . Note in particular that both  $u^+$  and  $u^-$  are positive. The modulus of  $u$ ,  $u \vee (-u)$ , is denoted by  $|u|$ , and satisfies  $|u| = u^+ + u^-$ . The following lemma is easily verified:

LEMMA 1.1.1. If  $V$  is a partially ordered vector space, then the following are equivalent:

- (a)  $u \vee v$  exists for every  $u, v \in V$
- (b)  $u \wedge v$  exists for every  $u, v \in V$
- (c)  $u^+$  exists for every  $u \in V$
- (d)  $u^-$  exists for every  $u \in V$
- (e)  $|u|$  exists for every  $u \in V$ .

If  $V$  satisfies any (and hence all) of (a) - (e), then  $V$  is termed a vector lattice. A partially ordered vector space  $V$  is termed  $\sigma$ -complete if for every countable set  $\{u_i\}_{i \in \mathbb{Z}^+}$  which is bounded above, i.e. there is  $v \in V$  with  $v \geq u_i$  for  $i=1,2,\dots$ , the supremum exists.  $V$  is complete if any subset that is bounded above has a supremum. Note that we do not require that  $V$  be a lattice, however if  $V$  is positively generated and  $(\sigma-)$ complete then it will be a lattice. Such a  $V$  will be termed a  $(\sigma-)$ complete lattice.

Two other properties that we wish to name are these;

(1)  $V$  has the Riesz decomposition property (R.D.P.), if whenever  $u, v, w \in V_+$  and  $0 \leq u \leq v + w$ , we can find  $u_1, u_2 \in V$  such that  $0 \leq u_1 \leq v$ ,  $0 \leq u_2 \leq w$ , and  $u_1 + u_2 = u$ .

(2)  $V$  has the Riesz separation property (R.S.P.), if whenever  $u_1, u_2 \leq w_1, w_2$ , there is a  $v \in V$  satisfying  $u_1, u_2 \leq v \leq w_1, w_2$ .

These two properties are equivalent. In fact we have:

LEMMA 1.1.2. If  $V$  is a partially ordered vector space, then

$(a) \Rightarrow (b) \Leftrightarrow (c)$ .

(a)  $V$  is a vector lattice.

(b)  $V$  has the R.D.P.

(c)  $V$  has the R.S.P.

A vector subspace  $W$  of a partially ordered vector space  $V$ , is an order ideal if  $w \in W_+$  and  $0 \leq v \leq w$  imply  $v \in W$ . A positively generated order ideal is termed an ideal.

An order unit for a partially ordered vector space  $V$ , is a positive element  $e$ , such that for each  $v \in V$  there is a  $\lambda \in R_+$  such that  $\lambda e \gg v \gg -\lambda e$ . An approximate order unit is a net  $\{e_\gamma : \gamma \in \Gamma\}$  in  $V_+$  such that:

(a)  $\gamma_1 \geq \gamma_2 \Rightarrow e_{\gamma_1} \gg e_{\gamma_2}$

(b) For each  $v \in V$  there is a  $\gamma(v) \in \Gamma$  and a  $\lambda(v) \in R_+$  with  $\lambda(v)e_{\gamma(v)} \gg v \gg -\lambda(v)e_{\gamma(v)}$ .

Thirdly a weak order unit for  $V$  is an element  $p \in V_+$  such that from the assumption  $u \leq v, p \Rightarrow u \leq 0$ , it follows that  $v \leq 0$ .

Examples of these three concepts abound in spaces that occur in analysis. If  $\Omega$  is a compact Hausdorff space, and  $C(\Omega)$  is ordered by  $f \geq 0$  if and only if  $f(w) \geq 0$  for all  $w \in \Omega$ , then the constantly one function is an order unit for  $C(\Omega)$ . In future such an ordering will be termed a pointwise partial ordering.

Consider the space of all continuous functions on a locally compact Hausdorff space  $\Sigma$ , that vanish at infinity. Give this space the pointwise order, then the family of all positive functions of supremum norm at most one provides an example of an approximate order unit.

If the space is e.g. the reals, then it is possible to construct a positive continuous function on  $\Sigma$  vanishing at infinity but at no point of  $\Sigma$ . Such a function will be a weak order unit for this space. It is easily verified that if  $e$  is an order unit for  $V$ , then it is also a weak order unit; and that any net constantly  $e$  is an approximate order unit.

A base for the wedge  $V_+$  in  $V$  is a convex subset  $B$  of  $V$  such that:

- (a) If  $v \in V_+$ ,  $\exists \lambda \in R_+$  and  $b \in B$  with  $v = \lambda b$ .
- (b) If  $\lambda, \mu \in R_+ \setminus \{0\}$  and  $a, b \in B$  are such that  $\lambda a = \mu b$ , then  $\lambda = \mu$  and  $a = b$ .

An example of such a space is the vector space  $M(\Omega)$  of all Radon measures on a compact Hausdorff space  $\Omega$ , with the usual partial ordering. The probability measures,  $P(\Omega)$ , form a base for this cone.

If the order defined by  $V_+$  makes  $V$  a lattice, then  $B$  is termed a simplex.

From now on we shall deal solely with partially ordered Banach spaces,  $X, Y$ . We make from the start the assumption that the positive wedge is closed, i.e. if  $x_n \geq 0$  and  $x_n \rightarrow x$  in norm then  $x \geq 0$ . This will be assumed for all spaces denoted by the symbols  $X$  and  $Y$  that occur hereafter. Note that this necessarily means that all such spaces are Archimedean.

Historically, the first special spaces to be considered were the  $L$ -spaces and  $M$ -spaces studied by Kakutani in [1] and [2]. A Banach lattice is a lattice ordered Banach space,  $X$ , that satisfies  $|x| \leq |y| \implies \|x\| \leq \|y\|$ . If also the norm is additive on the positive cone, i.e.  $x, y \geq 0 \implies \|x + y\| = \|x\| + \|y\|$ , then  $X$  is termed an  $L$ -space. If instead we have

$\|x \vee y\| = \max\{\|x\|, \|y\|\}$  whenever  $x, y \geq 0$  then  $X$  is termed an  $M$ -space. Kakutani proved the following representation theorems:

THEOREM 1.1.3. If  $X$  is an  $L$ -space, then there is a locally compact Hausdorff space  $\Sigma$  and a positive regular Borel measure  $\mu$  defined on the  $\sigma$ -field,  $S$ , of Borel subsets of  $\Sigma$ , such that  $X$  is isometrically order-isomorphic to  $L^1(\Sigma, S, \mu)$ . If  $X$  has a weak order unit then  $\Sigma$  may be chosen to be compact.

THEOREM 1.1.4. If  $X$  is an  $M$ -space, there is a compact Hausdorff space  $\Omega$  such that  $X$  is isometrically order-isomorphic to a closed vector sublattice of  $C(\Omega)$ . If  $X$  has an order unit then we can find  $\Omega$  such that  $X$  is isometrically order-isomorphic to all of  $C(\Omega)$ .

Results similar to these were also obtained by Nakano in [1], [3], and [4]. Theorem 1.1.3 was improved by Fullerton [1], and Cunningham [1] has given a non order theoretic characterisation of the spaces of type  $L^1(\mu)$ . He has also in [2] and [3] considered a similar problem for the space  $C(\Omega)$ , with rather less satisfactory results.

The spaces of interest to us arise from defining a norm in terms of some of the order properties. We shall see later that they do in fact provide us with generalisations of  $L$ - and  $M$ -spaces.

If we look at the space  $C(\Omega)$ , for  $\Omega$  compact Hausdorff, then  $\sup\{|f(w)| : w \in \Omega\} \leq 1$  if and only if  $-1_\Omega \leq f \leq 1_\Omega$ , where  $1_\Omega$  denotes the function that is constantly one on  $\Omega$ .

This observation motivates the following:

PROPOSITION 1.1.5. Let  $V$  be an Archimedean ordered vector space,

and  $e$  an order unit for  $V$ . The expression

$$\|x\| = \inf \{ \lambda > 0 : -\lambda e \leq x \leq \lambda e \}$$

defines a norm on  $X$ .

Such a norm is termed an order unit norm. The second result arises in a similar way from observation of the space  $M(\Omega)$ .

PROPOSITION 1.1.6. Let  $V$  be a positively generated partially ordered vector space for which the positive cone  $V_+$  has a base  $B$ . If  $S = \text{co}(B \cup -B)$  is radially compact, then the expression

$$\|x\| = \inf \{ \lambda : x \in \lambda S \}$$

is a norm on  $V$ . The closed unit ball of  $V$  for this norm is  $S$ , and  $V$  is complete if  $S$  is compact for some topology.

This norm is termed a base-norm. These two norms interact in an extremely satisfactory manner. To see this we need a definition.

DEFINITION 1.1.7. If  $X$  is a partially ordered Banach space, then the wedge  $\{f \in X^* : f(x) \geq 0 \text{ for all } x \in X_+\}$ , in the Banach dual  $X^*$  of  $X$ , is termed the dual wedge to  $X_+$ , and is denoted by  $X_+^*$ .  $X$  will always be considered to be ordered by this wedge, so that  $(X^*)_+ = X_+^*$ .

We also take the opportunity to define, if  $\alpha \in \mathbb{R}_+$ ,  $X_\alpha$  to be the closed ball of radius  $\alpha$ , and centre the origin, in  $X$ . Also  $X_{\alpha+} = X_\alpha \cap X_+$ ;  $X_{\alpha+}^* = X_\alpha^* \cap X_+^*$ .

We can now state the results that we want.

THEOREM 1.1.8. If  $X$  is normed by the order unit norm induced by the order unit  $e$ , then the base-norm on  $X^*$  induced by the base  $B = \{f \in X^* : f \geq 0, f(e) = 1\}$  coincides with the usual

norm as a dual space. Also the base  $B$  is compact for the weak\*-topology of  $X^*$ .

Conversely if  $X$  has a closed cone, and  $X^*$  is base-normed by a weak\*-compact base, then the norm in  $X$  is an order unit norm.

COROLLARY 1.1.9. Let  $K$  be a compact convex set in a locally convex Hausdorff topological vector space. If  $A(K)$  denotes the space of all continuous affine functions on  $K$  with the supremum norm and pointwise ordering, then the set  $\{f \in A(K)^* : f \geq 0, f(1_K) = 1\}$ , with the weak\*-topology, is affinely homeomorphic to  $K$ .

THEOREM 1.1.10. If  $X$  is base-normed by a base  $B$  then  $X^*$  is order unit normed by the functional that is identically one on  $B$ . Conversely if  $X_+$  is closed and  $X^*$  is order unit normed, then  $X$  is base-normed.

For various purposes, the order unit normed spaces are not sufficient for our needs. In order to study compact simplexes in detail, Effros in [1] and [2], and with Gleit in [1], has studied simplex spaces. These are partially ordered Banach spaces with closed cones, whose dual is an  $L$ -space. Such spaces always have the R.D.P. As Kakutani showed that the dual of an  $M$ -space is an  $L$ -space, it follows that each  $M$ -space is a simplex space. Not all simplex spaces are  $M$ -spaces, and not all have order units.

In order to provide a more intrinsic definition of a simplex space, and to complete our duality results, we need one more special type of space.

PROPOSITION 1.1.11. Let  $V$  be a partially ordered vector space with an approximate order unit,  $\{e_\gamma : \gamma \in \Gamma\}$ , such that  $S = \bigcup_{\gamma \in \Gamma} [-e_\gamma, e_\gamma]$  is linearly bounded. Then the expression

$$\|x\| = \inf \{ \lambda : x \in \lambda S \}$$

defines a norm on  $V$ .

Such a norm will be termed an approximate order unit norm, and abbreviated to a.o.u.-norm. We can now state:

THEOREM 1.1.12.  $X$  is a.o.u.-normed iff  $X^*$  is base-normed.

COROLLARY 1.1.13.  $X$  is a simplex space iff  $X$  is a.o.u.-normed and has the R.W.D.P.

The next result shows that the circle of duality results involving order unit-, base-, and a.o.u.-normed spaces is in fact closed.

PROPOSITION 1.1.14. If  $X^*$  is a.o.u.-normed, then it is order unit normed.

Order unit norms were first used by Grosberg and Krein, [1]. Essentially, the first of our results to be proved was Theorem 1.1.8 by Edwards [1] and Ellis [2]. Theorem 1.1.10 is due to Ellis [2] and Theorem 1.1.12 and the two following results to Ng [1]. These last results were also obtained by Asimow, [1], in a slightly different form. It is also now possible for us to give representation theorems for order unit normed and a.o.u.-normed spaces, but we first need some more definitions.

The term compact convex set will be taken to mean a compact convex subset of a locally convex Hausdorff topological vector space, ( hereafter referred to as an l.c.s. ). A face of a compact convex set  $K$  is a convex subset  $F$  such that  $\lambda k_1 + (1 - \lambda)k_2 \in F$

for  $k_1, k_2 \in K$  and  $0 < \lambda < 1$  together imply  $k_1, k_2 \in F$ . If  $\{k\}$  is a face of  $K$ , then  $k$  is termed an extreme point of  $K$ . The set of all extreme points of  $K$  is denoted by  $\partial_e K$ .

The Krein-Milman theorem assures us that if  $K$  is such a compact convex set then  $\partial_e K \neq \emptyset$ . It further tells us that  $K = \overline{\text{co}}(\partial_e K)$ . An alternative way of stating this is to say that each point  $k$  of  $K$  can be represented by a Borel measure  $\mu$  supported by  $\overline{\partial_e K}$ , i.e.  $a(k) = \int a d\mu$  for any  $a \in A(K)$ . If  $K$  is metrizable then Choquet's theorem tells us that we can replace  $\overline{\partial_e K}$  by  $\partial_e K$ . In general the Bishop-de Leeuw theorem gives us a rather unsatisfactory generalisation of this. We shall not require these last two results, but an account of them may be found in Alfsen [3].

If  $P$  is a cone in an l.c.s.  $E$ , then a cap  $C$  of  $P$  is any compact convex subset of  $P$  such that  $P \setminus C$  is convex. If further  $P = \bigcup_{n=1}^{\infty} nC$  then  $C$  is a universal cap. We shall also term  $C$  a universal cap if it is affinely homeomorphic to a universal cap of some cone. A ray of the cone  $P$  is a set  $\{\lambda r : \lambda \in R_+\}$  for some  $r \in P$ . If for any two rays of  $P$ ,  $R_1$  and  $R_2$ , such that  $R_1 + R_2 = R$  it follows that  $R_1 = R_2 = R$ , then  $R$  is an extremal ray. The set of all extremal rays of  $P$  will be denoted by  $\partial_e P$ . If  $C$  is a universal cap of  $P$  then the extremal rays of  $P$  are precisely the sets  $\{\lambda r : \lambda \in R_+\}$  for  $r$  a non-zero extreme point of  $C$ . We use this remark to justify letting  $\partial_e C$  denote the set of all non-zero extreme points of  $C$ .

PROPOSITION 1.1.15. Any order unit normed space  $A$  is isometrically order isomorphic to the space of all continuous affine functions,  $A(K)$ , on the weak\*-compact convex set  $K = \{f \in A^* : f \geq 0, \|f\| = 1\}$ .

Any a.o.u.-normed space  $A$  is isometrically order isomorphic

to the space of all continuous affine functions vanishing at 0,  $A_0(C)$ , on the weak<sup>\*</sup>-compact cap  $C = \{f \in A^* : f \geq 0, \|f\| \leq 1\}$  of  $A_+^*$ .

Given a compact convex set,  $K$ , and an extreme point  $k_0$  of  $K$ , it is natural to ask whether or not  $K$  is affinely homeomorphic to a universal cap, with  $k_0$  corresponding to the origin. One characterisation is clearly that the space of all continuous affine functions on  $K$  that vanish at  $k_0$  be a.o.u.-normed. Another is given by Asimow in [3]. If the set is a simplex, then any extreme point has this property, so that the simplex spaces of Effros are simply the spaces,  $A_0(C)$ , of all continuous affine functions vanishing at some extreme point of some compact simplex  $C$ . In particular we note that if  $A$  is a simplex space then  $\{f \in A^* : f \geq 0, \|f\| \leq 1\}$  is a weak<sup>\*</sup>-compact simplex.

Before we leave these special spaces, we state some results on bounded linear operators between certain of them. The most general of these that we require is the following:

PROPOSITION 1.1.16. Let  $C$  be a universal cap,  $X$  a Banach space, and  $T$  a bounded linear operator from  $X$  into  $A_0(C)$ . Then there is an affine map  $\tau$  of  $C$  into  $X^*$ , vanishing at the origin and continuous for the weak<sup>\*</sup>-topology of  $X^*$ , such that

$$(1) \quad (Tx)(k) = (\tau k)(x) \quad (x \in X, k \in C)$$

$$(2) \quad \|T\| = \sup \{\|\tau k\| : k \in C\}$$

Conversely if such a  $\tau$  is given, then (1) defines a bounded linear operator from  $X$  to  $A_0(C)$  with norm defined by (2).  $T$  is compact iff  $\tau$  is continuous for the norm topology of  $X^*$ .

If  $X$  is partially ordered by a closed cone, then  $T \geq 0$  iff  $\tau \geq 0$ .

Proof. All of this, except for the last remark, is proved

(essentially) in Dunford and Schwartz, [1]. For this last remark simply observe:

$$\begin{aligned}
 T \geq 0 &\iff T_x \geq 0 & (\forall x \geq 0) \\
 &\iff (T_x)(k) \geq 0 & (\forall x \geq 0, \forall k \in C) \\
 &\iff (\tau k)(x) \geq 0 & (\forall x \geq 0, \forall k \in C) \\
 &\iff \tau k \geq 0 & (\forall k \in C) \\
 &\iff \tau \geq 0.
 \end{aligned}$$

Specialising slightly we obtain:

COROLLARY 1.1.17. If  $A_0(C_1)$  and  $A_0(C_2)$  are a.o.u.-normed spaces then  $T: A_0(C_1) \rightarrow A_0(C_2)$  is positive and of norm at most 1 iff  $\tau(C_2) \subset C_1$ . Also  $T: A(K_1) \rightarrow A(K_2)$  is positive and  $\|T\|_{K_1} = 1_{K_1}$  iff  $\tau(K_2) \subset K_1$ .

When dealing with order unit normed spaces  $A_1$  and  $A_2$ , the extreme points of the convex set  $\mathcal{A}(A_1, A_2)$  of positive linear operators which preserve the distinguished order unit are of interest. These we term extreme positive operators. One case in which these operators are completely characterised is the following due to A. and C. Ionescu-Tulcea, [1], Phelps [1], and Ellis [1].

THEOREM 1.1.18. Let  $X$  and  $Y$  be compact Hausdorff spaces. Then the following are equivalent:

- (a)  $T$  is an extreme point of  $\mathcal{A}(C(X), C(Y))$ .
- (b)  $T$  is an algebra homomorphism.
- (c)  $T$  is a lattice homomorphism.
- (d)  $\tau(Y) \subset X$ .

(We use here the identification of  $X$  with the extreme points of the base  $P(X)$  of  $C(X)_+^*$ .)

This result has been extended slightly by Lazar [1].

THEOREM 1.1.19. Let  $K$  be a compact simplex,  $\Omega$  a compact metric space. Then the following are equivalent:

- (a)  $T$  is an extreme point of  $\mathcal{A}(C(\Omega), A(K))$ .
- (b)  $T(f \vee g)$  is the supremum of  $Tf$  and  $Tg$  in  $A(K)$ .
- (c)  $\tau(\partial_e K) \subset \Omega$ .

Results similar in nature to these, characterising extreme points of certain convex sets of linear operators may be found in Blumenthal, Lindenstrauss and Phelps [1]; Bonsall, Lindenstrauss and Phelps [1]; and Morris and Phelps [1]. Related results for sets of linear functionals may be found in Buck [1] and Hayes [1].

If  $K_1$  and  $K_2$  are simplexes and  $T \in \mathcal{A}(A(K_1), A(K_2))$  then Jellet, [3], has defined  $T$  to be an  $R$ -homomorphism if whenever  $a, b \in A(K_1)$ ,  $f \in A(K_2)$  and  $Ta, Tb \leq f$ , there exists  $c \in A(K_1)$  such that  $a, b \leq c$  and  $Tc \leq f$ . These he characterised by:

THEOREM 1.1.20. Let  $K_1, K_2$  be compact simplexes and  $T \in \mathcal{A}(A(K_1), A(K_2))$ , then the following are equivalent:

- (a)  $T$  is an  $R$ -homomorphism.
- (b)  $\tau(\partial_e K_2) \subset \partial_e K_1$ .

Note that these do not exhaust the extreme points of  $\mathcal{A}(A(K_1), A(K_2))$ , see Jellet [3], even if  $A(K_2)$  is replaced by  $C(\Omega)$ , see Lazar [1].

### 1.2. Normality and positive generation.

Given a Banach space with a (closed) wedge, there are two properties of the wedge in which we are interested. These are, very roughly, whether or not the wedge is wide enough or narrow enough to be of use. In fact what we want is that the space be positively generated, and that there be a neighbourhood base of the origin consisting of sets that contain all order-intervals with end points in the set.

The first of these properties actually implies rather more, that the space is boundedly generated. I.e. that there is a constant  $M$  such that  $X_+ \subset X_{M+} - X_{M+}$ . We shall see that these two properties are mutually dual, i.e.  $X$  has one property if and only if  $X^*$  has the other. This result can be made rather exact, but first we need some more definitions.

The wedge  $X_+$  in  $X$  is C-generating if for all  $x \in X$  there are  $x^+, x^- \in X_+$  with  $x = x^+ - x^-$  and  $\|x^+\| + \|x^-\| \leq C \|x\|$ .  $X_+$  is C-normal if  $x \leq y \leq z$  implies  $\|y\| \leq C \max \{ \|x\|, \|z\| \}$ .  $X_+$  is normal if there is a neighbourhood base  $\{U_n\}$  of the origin of  $X$  such that  $x, z \in U_n$  and  $x \leq y \leq z$  imply  $y \in U_n$ .

It is easily seen that  $X_+$  is normal if and only if it is C-normal for some  $C$ , and is then certainly a cone. It is generating if and only if it is C-generating for some  $C$ . The basic theorem is the following:

THEOREM 1.2.1. Let  $C$  be a real constant. Then:

(a)  $X_+$  is C-normal iff  $X_+^*$  is C-generating.

(b)  $X_+$  is  $(C + \epsilon)$ -generating for all  $\epsilon > 0$  iff  $X_+^*$

is C-normal.

The first part of this was originally proved by Grosberg and Krein in [1], and the second by Ellis [2]. The original

proofs of both of these used properties of the special spaces considered in the last section. A short proof not depending on these has been given by Ng, [4].

There are various other related properties in use. One of the most important is the following due to Asimow.

$X$  is  $(\alpha, n)$ -directed if given  $x_1, x_2, \dots, x_n \in X_+$ , there is  $x \in X_+$  with  $x \geq x_1, x_2, \dots, x_n$ .  $X$  is approximately  $(\alpha, n)$ -directed if it is  $(\alpha + \varepsilon, n)$ -directed for all  $\varepsilon > 0$ .  $X$  is  $(\alpha, n)$ -additive if  $x_1, x_2, \dots, x_n \in X_+$  implies  $\sum_{i=1}^n \|x_i\| \leq \alpha \left\| \sum_{i=1}^n x_i \right\|$ .

**THEOREM 1.2.2.** Let  $\alpha$  be a positive real number and  $n$  a positive integer.

- (a)  $X$  is  $(\alpha, n)$ -additive iff  $X^*$  is  $(\alpha, n)$ -directed.
- (b)  $X$  is approximately  $(\alpha, n)$ -directed iff  $X^*$  is  $(\alpha, n)$ -additive.

The second part of this was proved by Asimow in [3]. Ng, in [2], gives an alternative proof, and states the first part without proof. A proof in a slightly more general context is contained in Chapter 2.

We now define  $X$  to be  $\alpha$ -directed if it is  $(\alpha, n)$ -directed for all  $n$ , and approximately  $\alpha$ -directed if it is approximately  $(\alpha, n)$ -directed for all  $n$ . If  $X$  is  $(\alpha, n)$ -additive for all  $n$  we term it  $\alpha$ -additive.

These conditions are obviously very restrictive. An immediate consequence of Theorem 1.2.2 is the following:

**COROLLARY 1.2.3.** If  $\alpha$  is a positive real, then:

- (a)  $X$  is  $\alpha$ -additive iff  $X^*$  is  $\alpha$ -directed.
- (b)  $X$  is approximately  $\alpha$ -directed iff  $X^*$  is  $\alpha$ -additive.

We shall later want to know when the unit ball of  $X$  is

order bounded. This is clearly so if  $X$  is base-normed. In general we have:

PROPOSITION 1.2.4. The unit ball in  $X^*$  is order bounded iff  $X$  is  $\alpha$ -additive for some  $\alpha$ .

Proof. Clearly if  $\|x_0\| = \alpha$  and  $x_0 \geq x$  for all  $x \in X_+^*$ , then  $X^*$  is  $\alpha$ -directed, and hence  $X$  is  $\alpha$ -additive. Conversely if  $X$  is  $\alpha$ -additive, then  $X^*$  is  $\alpha$ -directed so that the family  $\{U(x) : \|x\| \leq 1\}$ , where  $U(x) = \{y : y \geq x, \|y\| \leq \alpha\}$ , has the finite intersection property. As the cone in  $X$  is weak\*-closed, the sets  $U(x)$  are weak\*-compact. It follows that  $\bigcap \{U(x) : \|x\| \leq 1\} \neq \emptyset$ . But if  $x_0$  belongs to this intersection, then  $x_0 \geq x$  for all  $x \in X_+^*$ , so that  $X_+^*$  is order bounded as stated.

COROLLARY 1.2.5. If  $X$  is  $\alpha$ -additive and positively generated, then  $X$  has an equivalent norm under which it is base-normed.

Proof. Let  $e$  be an upper bound for the unit ball of  $X^*$ . As  $X$  is positively generated,  $X_+^*$  is normal, so that

$$X_+^* \subset \{x : e \geq x \geq -e\} \subset X_C^* \quad (\text{for some } C \geq 1)$$

Thus  $X^*$  has an order unit norm equivalent to the given norm. The subdual norm on  $X$  will be a base-norm equivalent to that given.

There are some more properties of interest:

THEOREM 1.2.6. Let  $\alpha \geq 1$ , then the following are equivalent:

- (a) If  $x \in X$  and  $\|x\| < 1$ , there is  $y \geq x, 0$  with  $\|y\| < \alpha$ .
- (b)  $f, g \in X^*$  and  $0 \leq f \leq g$  imply  $\|f\| \leq \alpha \|g\|$ .

This is due to Ng [2]. A dual result will be given in Chapter 2. He also proved the following result there, and again a dual result may be found in Chapter 2.

THEOREM 1.2.7. Let  $\alpha \geq 1$ , then the following are equivalent:

- (a) If  $x \in X$  and  $\|x\| < 1$ , there is  $y \gg x, -x$  with  $\|y\| < \alpha$ .
- (b)  $f, g \in X^*$  and  $-g \leq f \leq g$  imply  $\|f\| < \alpha \|g\|$ .

Both of these properties were first used by Davies in [1] and [2], for the case  $\alpha = 1$ . The general duality of these pairs for  $\alpha = 1$  was separated and proved under weaker assumptions by Ng [5].

### 1.3. Order properties.

The first result that we want deals with the duality of the order properties in the case when  $X_+$  is normal and generating. To be precise we have:

THEOREM 1.3.1. Let  $X$  be a partially ordered Banach space with a closed, normal and generating cone. Then the following are equivalent:

- (a)  $X$  has the R.D.P.
- (b)  $X^*$  has the R.D.P.
- (c)  $X^*$  is a vector lattice.
- (d)  $X^*$  is a complete vector lattice.

Proof. The equivalence of (a), (b) and (c) was proved by Andô, [1]. An alternative proof may be found in Ng, [6]. (d) follows from (c) by a straightforward compactness argument, or immediately from (a) as in the proof of Theorem 2.4.1.

The rest of this section will be devoted to the order properties of some of the special spaces that we have defined. Firstly we consider the spaces  $A(K)$ , for  $K$  a compact convex set.

PROPOSITION 1.3.2. A compact convex set  $K$  is a simplex if and only

$A(K)$  has the R.D.P.

A compact simplex  $K$  is a Bauer simplex if  $\partial_e K$  is closed.

THEOREM 1.3.3. If  $K$  is a compact convex set then the following are equivalent:

- (a)  $K$  is a Bauer simplex.
- (b)  $A(K)$  is a lattice.
- (c)  $A(K)$  is isometrically order isomorphic to  $C(\partial_e K)$ .
- (d)  $K$  is affinely homeomorphic to  $P(\partial_e K)$ , with the weak\*-topology as the dual of  $C(\partial_e K)$ .

We can infer some results on the order structure of  $A(K)$  from purely Banach space considerations.

PROPOSITION 1.3.4. If  $K$  is a compact simplex and  $A(K)$  is a dual Banach space, then there is a compact hyperstonian space  $\Omega$  such that  $A(K)$  is isometrically order isomorphic to  $C(\Omega)$ .

In particular  $A(K)$  is a complete vector lattice.

Proof. The intersection of  $A(K)_+$  with the unit ball of  $A(K)$  is the order interval  $[0, 1_K] = (\frac{1}{2})1_K + (\frac{1}{2})[-1_K, 1_K]$   
 $(\frac{1}{2})1_K + (\frac{1}{2})A(K)_+$ ,

and hence is weak\*-closed. Then  $A(K)_+$  is weak\*-closed by the Krein-Smulian theorem. It follows that the subdual of  $A(K)$  can be ordered by a closed cone in such a way that the ordering on  $A(K)$  is the dual ordering. Then Theorem 1.3.1 tells us that  $A(K)$  is a lattice, so is a space  $C(\Omega)$  for  $\Omega$  compact Hausdorff. Dixmier's paper, [1], then tells us that  $\Omega$  is hyperstonian.

The last comment is direct from Theorem 1.3.1, or from the fact that a hyperstonian space is Stonian and Theorem 1.3.5.

It is of great interest to know something about the order completeness of the lattice of continuous functions on topological spaces.

A topological space is extremally disconnected if the closure of every open set is open. A compact extremally disconnected space is called Stonian. A topological space is quasi-extremally disconnected if the closure of every open  $F_\sigma$  is open.

THEOREM 1.3.5. Let  $\Omega$  be a topological space, and  $C(\Omega)$  the ordered vector space of all continuous functions on  $\Omega$ .

If  $\Omega$  is extremally disconnected then  $C(\Omega)$  is order complete; and if  $\Omega$  is quasi-extremally disconnected then  $C(\Omega)$  is  $\sigma$ -complete. Conversely if  $\Omega$  is completely regular and  $C(\Omega)$  complete, then  $\Omega$  is extremally disconnected; and if  $\Omega$  is normal and  $C(\Omega)$   $\sigma$ -complete, then  $\Omega$  is quasi-extremally disconnected.

In particular if  $\Omega$  is a compact Hausdorff space, then  $C(\Omega)$  is complete iff  $\Omega$  is Stonian.

This result is due to Nakano, [2]. For various purposes we need to know more about functions defined on compact simplexes. The main theorem in the subject is the following due to Edwards; [2]. An order theoretic proof may be found in Ellis [4].

THEOREM 1.3.6. Let  $K$  be a compact simplex,  $f : K \rightarrow [-\infty, \infty)$  an u.s.c. convex function, and  $g : K \rightarrow (-\infty, \infty]$  a l.s.c. concave function. If  $f(k) \leq g(k)$  for all  $k \in K$ , then there is an  $h \in A(K)$  such that  $f(k) \leq h(k) \leq g(k)$  for all  $k \in K$ .

This may be regarded as a special case of the following theorem of Lazar [2], a short proof of which has been given by Léger [1]. We first need some definitions.

Let  $X, Y$  be topological spaces. A map  $\bar{\Phi}: X \rightarrow 2^Y$  is termed lower semi-continuous if for every open set  $U \subset Y$ , the set  $\{x \in X: \bar{\Phi}(x) \cap U \neq \emptyset\}$  is open in  $X$ .

Let  $C$  be a convex subset of a linear space  $E$ , and  $F$  another linear space.  $\bar{\Phi}: C \rightarrow 2^F$  is termed affine if for every  $c \in C$ ,  $\bar{\Phi}(c)$  is a non-empty convex set, and

$$\lambda \bar{\Phi}(c_1) + (1-\lambda) \bar{\Phi}(c_2) \subset \bar{\Phi}(\lambda c_1 + (1-\lambda)c_2)$$

for all  $c_1, c_2 \in C$ ,  $0 < \lambda < 1$ .

THEOREM 1.3.7. Let  $K$  be a compact simplex,  $E$  a Fréchet space. Let  $\bar{\Phi}: K \rightarrow 2^E$  be affine and lower semi-continuous with  $\bar{\Phi}(k)$  closed for all  $k \in K$ . Then there exists a continuous affine selection for  $\bar{\Phi}$ ; i.e. a continuous affine function  $\varphi: K \rightarrow E$  with  $\varphi(k) \in \bar{\Phi}(k)$  for all  $k \in K$ .

We would like to mention at this point the fact that simplexes were originally of interest because they are precisely those compact convex sets for which maximal representing measures, in the Choquet-Bishop-de Leeuw theory, are unique. Their main interest now is that most results true for spaces of continuous functions on a compact Hausdorff space can also be proved spaces  $A(K)$ , for  $K$  a simplex. In comparison with this case, very little is known about  $A(K)$  for  $K$  a general compact convex set.

The final result that we require is:

PROPOSITION 1.3.8. Let  $C$  be a universal cap.  $A_0(C)$  has the R.D.P. if and only if  $C$  is a simplex.

There is no easy characterisation of when  $A_0(C)$  is a lattice. We do know a little of the order properties of this space when it is a lattice, but we defer this result until we have need of it.

## Chapter II

Spaces of bounded linear operators between partially ordered  
Banach spaces.

In this chapter we study spaces of bounded linear operators between partially ordered Banach spaces. The first section deals with the normality of  $L(X, Y)_+$  and the second with conditions that the space be positively generated. In the third section we look at some special cases, and in the fourth deal with the order structure of the space. Finally we use the opportunity to state some properties of the projective tensor product  $X \otimes_Y Y$  which can be deduced from the properties of these spaces of linear operators. To avoid trivial cases we assume  $X_+, Y_+ \neq \{0\}$ .

### 2.1. Normality.

$L(X, Y)$  will denote the Banach space of all bounded linear operators from  $X$  to  $Y$ , with the usual norm. This space will be provided with the positive wedge of all those operators  $T$  such that  $T(X_+) \subset Y_+$ . It is easily seen that this wedge is closed. We can determine exactly when  $L(X, Y)_+$  is normal.

THEOREM 2.1.1. The positive wedge in  $L(X, Y)$  is normal if and only if  $X$  is positively generated and  $Y_+$  is normal.

Proof. Suppose  $X$  is  $C$ -generated and  $Y_+$  is  $D$ -normal. Let

$S, T, U \in L(X, Y)$ ,  $S \leq T \leq U$  and that  $\|S\|, \|U\| \leq 1$ . If  $\|x\| \leq 1$ ,  $\exists x^+, x^- \geq 0$  such that  $x = x^+ - x^-$ ,  $\|x^+\| + \|x^-\| \leq C$ . As  $\|Tx\| \leq \|Tx^+\| + \|Tx^-\|$ , we see that

$$\|T\| \leq \sup \left\{ \|Tx^+\| + \|Tx^-\| : x^+, x^- \geq 0, \right. \\ \left. \|x^+\| + \|x^-\| \leq C \right\}$$

$$\begin{aligned}
&\leq C \sup \{ \|Ty\| : y \geq 0, \|y\| \leq 1 \} \\
&\leq C D \sup \{ \max \{ \|Sy\|, \|Uy\| \} : y \geq 0, \|y\| \leq 1 \} \\
&\leq C D \max \{ \|S\|, \|U\| \}.
\end{aligned}$$

Hence  $L(X, Y)_+$  is CD-normal.

Suppose, conversely, that  $L(X, Y)_+$  is A-normal. Choose  $y \in Y_+$ , with  $\|y\| = 1$ . If  $f, g, h \in X^*$  and  $f \leq g \leq h$ , let  $F : x \mapsto f(x)y$ , etc. As  $F \leq G \leq H$ ,  $\|G\| \leq A \max \{ \|F\|, \|H\| \}$ . But  $\|f\| = \|F\|$ , etc., so that  $X_+^*$  is A-normal. It follows that  $X$  is  $(A + \varepsilon)$ -generated for all  $\varepsilon > 0$ .

To see that  $Y_+$  is normal, let  $f$  be a non-zero, positive bounded linear functional on  $X$  (such exist since  $X_+$  is closed). Let  $x_0 \in X_+$ ,  $\|x_0\| = 1$  and  $f(x_0) = 1$  (if necessary using a multiple of  $f$ ). Suppose  $s \leq t \leq u$  with  $s, t, u \in Y$ , and define  $S : x \mapsto f(x)s$ , etc. Clearly  $S \leq T \leq U$  and  $\|S\| = \|f\| \|s\|$  etc. We now clearly see that if  $L(X, Y)_+$  is A-normal, then so is  $Y_+$ .

We can make some comments on the constants involved from the proof. The situation is especially satisfactory if either  $Y_+$  is 1-normal, or if  $X$  is  $(1 + \varepsilon)$ -generated for all  $\varepsilon > 0$ .

COROLLARY 2.1.2. Let  $Y_+$  be 1-normal, then  $X$  is  $(C + \varepsilon)$ -generated for all  $\varepsilon > 0$  if and only if  $L(X, Y)_+$  is C-normal.

Let  $X$  be  $(1 + \varepsilon)$ -generated for all  $\varepsilon > 0$ , then  $Y_+$  is C-normal if and only if  $L(X, Y)_+$  is C-normal.

In general we cannot be so precise about the constants involved. We can however obtain some inequalities.

PROPOSITION 2.1.3. Suppose  $X$  is  $(C + \varepsilon)$ -generated for all  $\varepsilon > 0$ ,  $Y_+$  is D-normal, and  $L(X, Y)_+$  is A-normal. Suppose also that all

these constants are the best possible. Then  $CD \geq A \geq C, D$ .

Proof. The proof is contained in that of Theorem 2.1.1.

It does not seem possible, in general, to give an exact statement of the relationship between the constants  $A, C$  and  $D$  that we have here. We are not convinced that such a simple relationship exists between the spaces.

However in all the situations in which the constants can be calculated, we do have  $A = CD$ . But in all such situations, the norms of the spaces  $X$  and  $Y$  interact in a very nice manner. It seems to us that this is the reason for equality in these cases. In general the norms of operators are difficult to compute exactly, making the study of the normality constants not easily performed.

We have one more result to state. It follows from Proposition 2.1.3 that  $A$  and  $CD$  tend to infinity together, as  $A \geq \sqrt{CD}$ . We can improve this slightly.

PROPOSITION 2.1.4. With the notation of Proposition 2.1.3,  
 $CD \geq A \geq \frac{1}{4} (C-2)(D-2)$ .

Proof. By assumption, for all  $\delta > 0$  there are  $s, t, u \in Y$  such that  $s \leq t \leq u$  and  $\|t\| \geq (D - \delta) \max \{\|s\|, \|u\|\}$ . Consider the norm of  $(t-s)$ , which satisfies  $0 \leq t-s \leq u-s$ .

$$\begin{aligned} \|t-s\| &\geq \|t\| - \|s\| \\ &\geq (D - \delta) \max \{\|s\|, \|u\|\} - \|s\| \\ &\geq (D - \delta)/2 (\|s\| + \|u\|) - \|s\| \\ &\geq (D-2 - \delta)/2 (\|s\| + \|u\|) \\ &\geq (D-2 - \delta)/2 \|u-s\|. \end{aligned}$$

Hence we can find  $x, y \in Y$  with  $0 \leq x \leq y$  and  $\|x\| \geq (D-2 - \delta)/2 \|y\|$ .

Similarly we can find  $f, g \in X^*$  with  $0 \leq f \leq g$  and  $\|f\| \geq (C-2-\delta)/2 \|g\|$ . Letting  $F : u \mapsto f(u)x$  and  $G : u \mapsto g(u)y$ , we easily see that  $0 \leq F \leq G$  and  $\|F\| \geq \frac{1}{4}(D-2-\delta)(C-2-\delta) \|G\|$ . We then clearly have  $A \geq \frac{1}{4}(C-2)(D-2)$  as claimed.

We can also obtain some estimates of the constants involved in the properties defined by Asimow.

PROPOSITION 2.1.5. Let  $X$  be  $(\alpha, n)$ -directed,  $Y$   $(\beta, n)$ -additive and  $D$ -normal. Then  $L(X, Y)$  is  $(\alpha \beta D, n)$ -additive.

Proof. Note firstly that if  $y \in Y$  then  $\|y\|_1 \leq \|y\| \leq D \|y\|_1$ , where  $\|y\|_1$  is the norm  $\sup \{ |f(y)| : f \in Y_{1+}^* \}$ .

If  $S_i$ , for  $1 \leq i \leq n$ , are positive operators in  $L(X, Y)$ , then we have:

$$\begin{aligned} \sum \|S_i\| &= \sum \sup \{ \|S_i x\| : \|x\| \leq 1 \} \\ &\leq \sum^D \sup \{ \|S_i x\|_1 : \|x\| \leq 1 \} \\ &\leq \sum^D \sup \{ \sup \{ f(S_i x) : f \in Y_{1+}^* \} : \|x\| \leq 1 \} \\ &\leq \alpha^D \sup \{ \sum \sup \{ f(S_i x) : f \in Y_{1+}^* \} : \|x\| \leq 1 \} \\ &\leq \alpha^D \sup \{ \sum \|S_i x\|_1 : \|x\| \leq 1 \} \\ &\leq \alpha \beta^D \sup \{ \|\sum S_i x\| : \|x\| \leq 1 \} \\ &= \alpha \beta^D \|\sum S_i\|. \end{aligned}$$

Even if we take  $Y_+$  to be 1-normal, we cannot assert that this result is the best possible.

COROLLARY 2.1.6. Let  $X$  be  $\alpha$ -directed,  $Y$   $\beta$ -additive and  $D$ -normal. Then  $L(X, Y)$  is  $(\alpha \beta^D)$ -additive.

Considering the two pairs of properties used by Ng, we still cannot obtain a precise result. However we can obtain the following inequalities.

PROPOSITION 2.1.7. Suppose that  $X$  satisfies  $\|x\| < 1 \Rightarrow \exists y \gg x, -x$  with  $\|y\| < \alpha$  and that  $\exists y' \gg x, 0$  with  $\|y'\| < \alpha'$ ; that  $Y$  satisfies  $s \gg t \gg -s \Rightarrow \|t\| \leq \beta \|s\|$  and that  $s' \gg t' \gg 0 \Rightarrow \|t'\| \leq \beta' \|s'\|$ ; and that  $L(X, Y)$  satisfies  $S \gg T \gg -S \Rightarrow \|T\| \leq \gamma \|S\|$ , and that  $S' \gg T' \gg 0 \Rightarrow \|T'\| \leq \gamma' \|S'\|$ . Suppose further that all these constants are the best possible, Then  $\alpha \beta \geq \gamma \geq \gamma' \geq \alpha' \beta'$ .

Proof. To see that  $\gamma \leq \alpha \beta$ , let  $S \gg T \gg -S$ . If  $\|x\| < 1$ ,  $\exists y \gg x, -x$  with  $\|y\| < \alpha$ . Then we have  $Sy \gg Tx \gg -Sy$ , so that  $\|Tx\| \leq \alpha \beta \|S\|$ . Taking the supremum over  $\|x\| \leq 1$ , we obtain  $\|T\| \leq \alpha \beta \|S\|$ , so that  $\gamma \leq \alpha \beta$ .

It is clear that  $\gamma \geq \gamma'$ . Suppose  $\delta > 0$ , and  $x, y \in Y$  with  $0 \leq x \leq y$  and  $\|x\| = (\beta' - \delta) \|y\|$ . Similarly there are  $f, g \in X^*$ ,  $0 \leq f \leq g$  and  $\|f\| = (\alpha' - \delta) \|g\|$ . Let  $F : u \mapsto f(u)x$  and  $G : u \mapsto g(u)y$  so that  $G \gg F \gg 0$ . Clearly  $\|F\| = (\alpha' - \delta)(\beta' - \delta) \|G\|$ , so that  $\gamma' \geq (\alpha' - \delta)(\beta' - \delta)$  for all  $\delta > 0$ . Hence  $\gamma' \geq \alpha' \beta'$  as claimed.

COROLLARY 2.1.8. With the notation of Proposition 2.1.7, if  $\beta = 1$  then  $\alpha = \gamma$ .

Proof. We have  $\gamma \leq \alpha$  by this last result. We show that  $-g \leq f \leq g$ , with  $f, g \in X^* \Rightarrow \|f\| \leq \gamma \|g\|$ , so that Theorem 1.2.7 will yield the result. Let  $y \in Y_+$ , with  $\|y\| = 1$ . Then  $-g \otimes y \leq f \otimes y \leq g \otimes y$ , where  $f \otimes y : x \mapsto f(x)y$  etc. By assumption  $\|f\| = \|f \otimes y\| \leq \gamma \|g \otimes y\| = \gamma \|g\|$ , so the result is complete.

## 2.2. Positive generation.

In this section we shall look at conditions for  $L(X, Y)$  to be positively generated. We cannot give a complete answer to this question, but we do provide certain conditions under which the space is so. Firstly we look at the necessary conditions.

PROPOSITION 2.2.1. If  $L(X, Y)$  is  $(C + \epsilon)$ -generated for all  $\epsilon > 0$ , then  $X_+$  is  $C$ -normal and  $Y$  is  $(C + \epsilon)$ -generated for all  $\epsilon > 0$ .

Proof. Let  $f \in Y_+^*$  with  $\|f\| = 1$ , and  $x, y, z \in X$  with  $x \leq y \leq z$ . Each of  $x \otimes f, y \otimes f, z \otimes f$  can be represented as bounded linear functionals on  $L(X, Y)$ , and

$$x \otimes f \leq y \otimes f \leq z \otimes f.$$

As  $L(X, Y)_+^*$  is  $C$ -normal,

$$\begin{aligned} \|y\| &= \|y \otimes f\| \\ &\leq C \max \{\|x \otimes f\|, \|z \otimes f\|\} \\ &= C \max \{\|x\|, \|z\|\}, \end{aligned}$$

so  $X$  is  $C$ -normal.

Similarly, let  $x \in X_+$  with  $\|x\| = 1$ , and  $f, g, h \in Y^*$  with  $f \leq g \leq h$ , so that

$$x \otimes f \leq x \otimes g \leq x \otimes h.$$

Again we have

$$\begin{aligned} \|g\| &= \|x \otimes g\| \\ &\leq C \max \{\|x \otimes f\|, \|x \otimes h\|\} \\ &= C \max \{\|f\|, \|h\|\}. \end{aligned}$$

Thus  $Y_+^*$  is  $C$ -normal, and  $Y$   $(C + \epsilon)$ -generated for all  $\epsilon > 0$ .

One of the cases in which the space is known to be

positively generated is when  $Y = C(\Omega)$ ,  $\Omega$  a Stonian space. Such a space is known to be a complete vector lattice, so the following theorem of Bonsall, [1], is of use to us.

**THEOREM 2.2.2. (BONSALL)** Let  $E$  be a real vector space, with  $E_+$  a wedge in  $E$ . Suppose  $P$  is a sublinear functional taking values in a complete vector lattice  $V$ , and  $Q$  a superlinear functional defined on  $E_+$  with values in  $V$  such that

$$\underline{Q(x) \leq P(x)} \quad (\forall x \in E_+).$$

Then there is a linear functional  $T$  from  $E$  into  $V$  such that:

$$\begin{aligned} \underline{T(x) \leq P(x)} & \quad (\forall x \in E) \\ \underline{Q(x) \leq T(x)} & \quad (\forall x \in E_+). \end{aligned}$$

The first result we prove is the dual of a result of Asimow [3], which was stated without proof for  $Y = \mathbb{R}$ , by Ng in [2].

**THEOREM 2.2.3.** Let  $\Omega$  be a Stonian space, then  $X$  is  $(\alpha, n)$ -additive if and only if  $L(X, C(\Omega))$  is  $(\alpha, n)$ -directed.

Proof. Let  $T_1, \dots, T_n : X \rightarrow C(\Omega)$  all have norm less than or equal to one. Let  $Q(x) = \sup \{ T_1(x_1) + \dots + T_n(x_n) : \sum_{i=1}^n x_i = x, x_i \geq 0 \}$  if  $x \geq 0$ .  $Q$  is well-defined, since

$$\begin{aligned} \sum_{i=1}^n T_i(x_i) & \leq \sum_{i=1}^n \|T_i(x_i)\| 1_\Omega \\ & \leq \sum_{i=1}^n \|x_i\| 1_\Omega \\ & \leq \alpha \|\sum_{i=1}^n x_i\| 1_\Omega \\ & \leq \alpha \|x\| 1_\Omega, \end{aligned}$$

and it is clear that  $Q$  is superlinear on  $X_+$ . Furthermore if  $P$  is the sublinear functional defined on  $X$  by mapping  $x$  to  $\alpha \|x\| 1_\Omega$ , then  $Q(x) \leq P(x)$  for all  $x \in X_+$ . By Theorem 2.2.2 there is  $S \in L(X, C(\Omega))$  such that

$$S(x) \geq Q(x) \quad (\forall x \in X_+)$$

$$S(x) \leq P(x) \quad (\forall x \in X),$$

(the latter inequality ensures that  $S$  is bounded.). It is clear from the definition of  $Q$  that  $S(x) \geq T_i(x)$  if  $1 \leq i \leq n$ , and  $x \geq 0$ . As  $\|S\| \leq \alpha$  the implication in one direction is proved.

Suppose, conversely, that  $x_1, \dots, x_n \in X_+$ . Let  $f_i \in X^*$ , with  $\|f_i\| = 1$ ,  $f_i(x_i) = \|x_i\|$ . If  $T_i : x \mapsto f_i(x)1_\Omega$ , then there is  $S \geq T_i$  ( $1 \leq i \leq n$ ), with  $\|S\| \leq \alpha$ . But then

$$\begin{aligned} \alpha \|\sum x_i\|1_\Omega &\geq S(\sum x_i) \\ &= \sum S(x_i) \\ &\geq \sum T_i(x_i) \\ &= \sum \|x_i\|1_\Omega. \end{aligned}$$

So  $\alpha \|\sum x_i\| \geq \sum \|x_i\|$ , and  $X$  is  $(\alpha, n)$ -additive.

COROLLARY 2.2.4. Let  $\Omega$  be a Stonian space, then  $X_+$  is normal iff  $L(X, C(\Omega))$  is positively generated.

We can prove similarly results dual to those of Theorems 1.2.6 and 1.2.7.

THEOREM 2.2.5. Let  $\alpha \geq 1$ ,  $\Omega$  a Stonian space. The following are equivalent:

- (a)  $x, y \in X$ ,  $0 \leq x \leq y \implies \|x\| \leq \alpha \|y\|$ .  
 (b)  $T \in L(X, C(\Omega))$ ,  $\|T\| \leq 1 \implies \exists S \geq T, 0$  with  $\|S\| \leq \alpha$ .

Proof. (a)  $\implies$  (b) follows again from Theorem 2.2.2, with

$$P(x) = \alpha \|T\| \|x\|, \text{ and } Q(x) = \sup \{T_y : 0 \leq y \leq x\}.$$

To prove (b)  $\implies$  (a), first note that this is true if  $C(\Omega) = \mathbb{R}$ , for then if (b) holds  $0 \leq x \leq y$ ,  $x, y \in X^{**} \implies \|x\| \leq \alpha \|y\|$ , (by 1.2.6.). As  $X \subset X^{**}$ , and  $X_+$  is closed (so that the original order on  $X$  coincides with the relative ordering as a subspace of  $X^{**}$ ), (a) is true.

In general choose  $w \in \Omega$ . If  $f \in X$  and  $\|f\| \leq 1$ , let  $F : x \mapsto f(x)1_\Omega$ . By (b)  $\exists G \geq F, 0$  with  $\|G\| \leq \alpha$ . Let  $g : x \mapsto G(x)(w)$ , then it is clear that  $g \in X^*$ ,  $g \geq f, 0$  and  $\|g\| \leq \alpha$ . Thus (b) is true if  $C(\Omega)$  is replaced by  $R$ , so that (a) holds.

THEOREM 2.2.6. Let  $\alpha \geq 1$ ,  $\Omega$  a Stonian space. The following are equivalent:

$$(a) \ x, y \in X, \ -y \leq x \leq y \implies \|x\| \leq \alpha \|y\|.$$

$$(b) \ T \in L(X, C(\Omega)), \ \|T\| \leq 1 \implies \exists S \geq T, -T \text{ with } \|S\| \leq \alpha.$$

Proof. Again (a)  $\implies$  (b) uses Theorem 2.2.2 with  $P(x) = \alpha \|T\| \|x\|$ , and  $Q(x) = \sup \{Ty : -x \leq y \leq x\}$ . The proof of (b)  $\implies$  (a) is almost identical with that for the corresponding part of Theorem 2.2.5.

One other case in which we can show that  $L(X, Y)$  is positively generated is when either  $X$  or  $Y$  is finite dimensional. The proof of this involves the following:

LEMMA 2.2.7. If  $X$  is a finite dimensional real vector space with a closed generating cone,  $X_+$ , then there exist closed, generating cones  $P_1, P_2$  such that  $P_1 \subset X_+ \subset P_2$  and  $P_1$  induces a lattice ordering on  $X$ .

Proof. Note firstly that as  $X$  is finite dimensional and  $X$  is closed,  $X_+$  must be normal. Also the interior of  $X_+$  is non-empty so that  $X$  has an order unit. As this same argument holds for  $X_+^*$ ,  $X_+$  has a base,  $B$ , by Theorem 1.1.10 and the equivalence of all norms on finite dimensional spaces. Certainly  $B$  is compact, and, if  $n$ -dimensional, we can find  $n+1$  affinely

independent extreme points of  $B$ . The convex hull of these points is an  $n$ -simplex,  $S$ . If  $P_1$  is the (closed) cone with base  $S$ , then clearly  $P_1 \subset X_+$ , and  $P_1 - P_1 = X$ .

To find  $P_2$ , first find  $P_1^* \subset X_+^*$  closed, generating and inducing a lattice ordering. Now let  $P_2 = P_1^{**} \supset X_+^{**} = X_+$ , (identifying  $X$  with  $X^{**}$ ). By Theorem 1.3.1  $P_2$  induces a lattice ordering on  $X$ . Clearly  $P_2$  is closed and generating so the result is complete.

PROPOSITION 2.2.8. If  $Y$  is positively generated and finite dimensional, and  $X_+$  is normal, then  $L(X, Y)$  is positively generated.

Proof. If  $Y$  is given the order induced by  $P_1$ , then  $L(X, Y)$  with the natural ordering is positively generated. But if  $S \geq T, 0$  for this ordering, then a fortiori  $S \geq T, 0$  for the original ordering. Thus  $L(X, Y)$  is positively generated as claimed.

PROPOSITION 2.2.9. If  $X$  is finite dimensional and  $Y$  is positively generated, then  $L(X, Y)$  is positively generated.

Proof. Consider  $X$  with the cone  $P_2$  containing  $X_+$ , generating  $X_+ - X_+$ , constructed as in Lemma 2.2.7. If  $T : X \rightarrow Y$ , let  $\{x_1, \dots, x_n\}$  lie each on one extreme ray of  $P_2$ , and together generate  $P_2$ . As  $Y$  is positively generated there is a  $y_i \geq Tx_i, 0$  for  $1 \leq i \leq n$ . Let  $Sx_i = y_i$ , and extend  $S$  to the whole of  $X_+ - X_+$  by linearity, and thence to the whole of  $X$  in any linear manner. Then we have  $Sx \geq Tx, 0$  for all  $x \in P_2$ , and  $S$  is bounded since  $X$  is finite dimensional. But certainly  $Sx \geq Tx, 0$  whenever  $x \in X_+$ , so that  $L(X, Y)$  is positively generated.

One other situation in which the space  $L(X, Y)$  is known

to be positively generated is when  $X$  is base-normed and  $Y$  order unit normed. We in fact have:

THEOREM 2.2.10. (ELLIS [3]). If  $X$  is base-normed and  $Y$  is order unit normed, then  $L(X,Y)$  is order unit normed.

Proof. Let  $e$  be the order unit in  $Y$ , and let  $E(x) = \|x\|e$  if  $x \in X_+$ .  $E$  is additive on the positive cone of  $X$ , so extends to a linear operator from  $X$  to  $Y$ . It is clear that  $E$  is bounded and positive. We claim that  $E$  is an order unit for  $L(X,Y)$  defining the operator norm.

If  $\|T\| \leq 1$ , then  $\|Tx\| \leq \|x\|$  so that  $-\|x\|e \leq Tx \leq \|x\|e$ . In particular if  $x \geq 0$  we have  $-E(x) \leq Tx \leq E(x)$ , so  $-E \leq T \leq E$ . On the other hand, Corollary 2.1.2 tells us that  $L(X,Y)_+$  is 1-normal, so that its unit ball is precisely the order-interval  $[-E, E]$ , thus completing the proof.

We can in fact show that  $L(X,Y)$  is positively generated in slightly more general circumstances.

THEOREM 2.2.11. Let  $\alpha \in R_+$ , suppose  $X$  is  $\alpha$ -additive and the unit ball of  $Y$  is order-bounded. Then  $L(X,Y)$  is positively generated.

Proof. Let  $T \in L(X,Y)$ . If  $x \geq 0$  define  $q(x) = \sup \left\{ \sum_{i=1}^n \|Ty_i\| : y_i \geq 0 \text{ and } \sum_{i=1}^n y_i = x, n=1,2,\dots \right\}$ .  $q$  is well-defined, for if  $y_i \geq 0$  and  $\sum y_i = x$ , then

$$\begin{aligned} \sum \|Ty_i\| &\leq \sum \|T\| \|y_i\| \\ &\leq \alpha \|T\| \|x\|. \end{aligned}$$

It is easily verified that  $q$  is superlinear. Putting  $p(x) = \alpha \|T\| \|x\|$ , we can apply Theorem 2.2.2 to obtain a linear

functional  $\varphi$  on  $X$  satisfying

$$\varphi(x) \leq p(x) \quad (\forall x \in X)$$

$$\varphi(x) \geq q(x) \quad (\forall x \in X_+).$$

Let  $e$  be an upper bound for the unit ball in  $Y$ . Define  $S : x \mapsto \varphi(x)e$ . It is clear that  $S \in L(X, Y)$ .

If  $x \geq 0$ , then  $Sx = \varphi(x)e \geq \|Tx\| e \geq Tx, 0$ . Thus  $S \geq T, 0$  and  $L(X, Y)$  is positively generated.

What we can now show is that this result cannot be improved.

In fact we have:

PROPOSITION 2.2.12. If  $L(X, Y)$  is positively generated whenever  $X$  is base-normed, then the unit ball of  $Y$  is bounded above.

Proof. Let  $X = Y \times R$ . Order  $X$  by the cone with base

$\{(y, 1) : \|y\| \leq 1\}$ , and give it the base-norm defined by this base. Let  $T \in L(X, Y)$  be the natural projection of  $X$  onto  $Y$ . By assumption there is  $S \in L(X, Y)$  with  $S \geq T, 0$ . Consider the map  $\pi$  of the unit ball of  $Y$  into  $Y$  defined by

$$\pi y = S(y, 1).$$

Clearly  $\pi y \geq y, 0$  for all  $y \in Y_+$ . Also  $\pi$  is affine. We claim that  $2\pi(0)$  is an upper bound for the unit ball in  $Y$ , for if  $y \in Y$ , then also  $-y \in Y_+$ , so we have:

$$\begin{aligned} 2\pi(0) &= \pi(y) + \pi(-y) \\ &\geq y + 0 \\ &= y. \end{aligned}$$

Dually we have:

PROPOSITION 2.2.13. If  $L(X, Y)$  is positively generated whenever  $Y$  is order unit normed, then  $X$  is  $\alpha$ -additive for some  $\alpha$ .

Proof. To see this, we use the fact that the map  $\pi : L(X, Y^*) \rightarrow L(Y, X^*)$  defined by

$$[(\pi T)(y)](x) = (Tx)(y)$$

is a linear isometry of  $L(X, Y^*)$  onto  $L(Y, X^*)$ . See e.g. Schatten [1]. This map is also an order isomorphism since:

$$\begin{aligned} \pi T \geq 0 &\iff (\pi T)(y) \geq 0 && (\forall y \in Y_+) \\ &\iff [(\pi T)(y)](x) \geq 0 && (\forall y \in Y_+, \forall x \in X_+) \\ &\iff (Tx)(y) \geq 0 && (\forall y \in Y_+, \forall x \in X_+) \\ &\iff Tx \geq 0 && (\forall x \in X_+) \\ &\iff T \geq 0. \end{aligned}$$

Suppose now, that the norm in  $X$  is not  $\alpha$ -additive for any  $\alpha$ . It follows from Proposition 1.2.4 that the unit ball in  $X^*$  is not order-bounded. Thus there is a base-normed space  $Y$  such that  $L(Y, X^*)$  is not positively generated. That is,  $L(X, Y^*)$  is not positively generated, whilst  $Y^*$  is order unit normed, completing the proof.

We can summarise these last two results as follows:

THEOREM 2.2.14. Let  $\mathfrak{X}, \mathfrak{Y}$  be classes of partially ordered Banach spaces with closed positive cones. Let  $\mathfrak{X}_1$  be the class of all those that are  $\alpha$ -additive for some  $\alpha$ , and  $\mathfrak{Y}_1$  those with order bounded unit balls. Suppose also that  $\mathfrak{X} \supset \mathfrak{X}_1$  and  $\mathfrak{Y} \supset \mathfrak{Y}_1$ .

(a) If  $X \in \mathfrak{X}_1$  and  $Y \in \mathfrak{Y}_1$  then  $L(X, Y)$  is positively generated.

(b) If  $L(X, Y)$  is positively generated whenever  $X \in \mathfrak{X}$  and  $Y \in \mathfrak{Y}$ , then  $\mathfrak{X} = \mathfrak{X}_1$  and  $\mathfrak{Y} = \mathfrak{Y}_1$ .

Specialising slightly we have:

COROLLARY 2.2.15. Let  $\mathcal{X}, \mathcal{Y}$  be classes of partially ordered Banach spaces with closed, normal generating cones. Let  $\mathcal{X}_1$  be the class of all such that are equivalent to base-normed spaces, and  $\mathcal{Y}_1$  those equivalent to order unit normed spaces. Suppose also that  $\mathcal{X} \supset \mathcal{X}_1$  and  $\mathcal{Y} \supset \mathcal{Y}_1$ .

(a) If  $X \in \mathcal{X}_1$  and  $Y \in \mathcal{Y}_1$ , then  $L(X, Y)$  is positively generated.

(b) If  $L(X, Y)$  is positively generated whenever  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ , then  $\mathcal{X} = \mathcal{X}_1$  and  $\mathcal{Y} = \mathcal{Y}_1$ .

We conclude the study of the positive generation of  $L(X, Y)$  with a discussion of those spaces  $Y$  such that  $L(X, Y)$  is positively generated whenever  $X$  is normal. To have some representation of the spaces involved, we limit ourselves to spaces which are also normal. By Proposition 2.2.12 such a  $Y$  is certainly equivalent to an order unit normed space. We know that  $Y$  has this property if  $Y$  is either finite dimensional, or if it is  $C(\Omega)$ , for  $\Omega$  Stonian. However there are other such spaces. To see this we use the following:

LEMMA 2.2.16. Let  $Y' \subset Y$ , and suppose that there is a bounded linear map  $P$  of  $Y$  into  $Y'$  such that  $y \geq 0 \implies Py \geq y$ . If  $L(X, Y)$  is positively generated for all  $X$  with normal positive cones, then the same holds for  $L(X, Y')$ .

Proof. If  $T \in L(X, Y')$ , then  $T$  can also be regarded as an element of  $L(X, Y)$ . Thus there exists  $S \geq T, 0$  with  $S \in L(X, Y)$ . If  $S' = P \circ S$  then  $S' \in L(X, Y')$ . Also if  $x \geq 0$  then  $S'x = P[Sx] \geq P[Tx] \geq Tx$  and also  $S'x \geq P[Sx] \geq P(0) = 0$ . Thus  $S' \geq T, 0$ , and  $L(X, Y)$  is positively generated.

EXAMPLE 2.2.17. Let  $\Omega$  be an infinite Stonian space, and let  $s$  be a non-isolated point in  $\Omega$ . Let  $\Omega_1 = \Omega \cup \{t\} \cup \{u\}$ , where  $t, u \notin \Omega$ . Let  $U \subset \Omega_1$  be open iff  $U \cap \Omega$  is open, so that  $\Omega_1$  is now Stonian. Let  $Y = \{f \in C(\Omega_1) : 2f(s) = f(t) + f(u)\}$ . As  $s$  is not isolated,  $Y$  is not a lattice, (but does have the R.D.P.). Define  $P : C(\Omega_1) \rightarrow Y$  as follows :

$$\begin{aligned} (Pf)(x) &= f(x) + f(t) + f(u) & (x \in \Omega) \\ &f(s) + f(t) + f(u) & (x = s \text{ or } x = u). \end{aligned}$$

Clearly  $P$  satisfies all the conditions of the Lemma, and  $P(C(\Omega_1)) \subset Y$ . It follows that  $L(X, Y)$  is positively generated whenever  $X$  is normal.

It seems likely that if  $Y$  is a space  $C(\Omega)$ , with  $\Omega$  compact and Hausdorff, and  $L(X, Y)$  is positively generated whenever  $X$  is normal, then  $\Omega$  is Stonian. We can prove this result in the case  $\Omega$  metrizable, and do so below. In fact it seems probable that if  $Y$  is separable and normal and has this property, then  $Y$  is finite dimensional, however we have not succeeded in proving this.

In the absence of the separability assumption, or the assumption that  $Y$  is a lattice, the example shows that little can be proved, at least in terms that are in current use. Probably there is some sense in which  $Y$  is "near" to a space  $C(\Omega)$ , for  $\Omega$  Stonian.

THEOREM 2.2.18. Let  $Y$  be separable, a lattice, and with  $Y_+$  normal. If, for all  $X$  with  $X_+$  normal,  $L(X, Y)$  is positively generated, then  $Y$  is finite dimensional.

Proof. We know that  $Y$  is equivalent to a space  $C(\Omega)$ , for  $\Omega$

metrizable and compact. If  $\Omega$  is not finite, let  $w_0 \in \Omega$  be in the closure of  $\Omega \setminus \{w_0\}$ . Define a sequence of open sets as follows:

Let  $w_1 \neq w_0$ , with  $d(w_1, w_0) < 1$ ; now define  $U_1 = \{w \in \Omega : d(w, w_1) < \frac{1}{2} d(w_0, w_1)\}$ . If  $w_n$  and  $U_n$  are defined choose  $w_{n+1} \neq w_0$  so that  $d(w_{n+1}, w_0) < \frac{1}{4} d(w_n, w_0)$ . Now let  $U_{n+1} = \{w \in \Omega : d(w, w_{n+1}) < \frac{1}{2} d(w_0, w_{n+1})\}$ .

It is easily seen that the sets  $U_k$  are non-empty, open and disjoint. The same holds for  $V_k = \bigcup_{n=0}^{\infty} U_{4kn+2^k-1}$ , ( $k=0,1,2,\dots$ ), and also  $w_0 \in \overline{V_k} \setminus V_k$  for each  $k$ .

Let  $X$  be the vector space of all real-valued bounded functions defined on  $\Omega$  of the form

$$f + \sum_{k=0}^{\infty} \lambda_k \chi_k \quad (\text{point-wise convergence}),$$

with  $f \in C(\Omega)$ , and  $\chi_k$  the characteristic function of  $V_k$ .

Note firstly that the decomposition of an arbitrary element of  $X$  into this form is unique, for if  $g$  is the function, and  $g = f + \sum_{k=0}^{\infty} \lambda_k \chi_k$ , then  $f(w_0) = g(w_0)$ . But then  $\lambda_k = \lim_{w \rightarrow w_0 \text{ in } V_k} (g(w) - f(w_0))$ , and  $f = g - \sum_{k=0}^{\infty} \lambda_k \chi_k$ , so that  $f$  and  $\lambda_k$  are well-defined.

From the uniqueness of this decomposition it follows also that  $X$  is a Banach space when given the supremum norm. Indeed, let  $g_n = f_n + \sum_{k=0}^n \lambda_k^n \chi_k$ , and suppose  $\|g_n - g_m\| < \varepsilon$ . It follows that:

$$(1) \quad |g_n(w_0) - g_m(w_0)| < \varepsilon.$$

$$(2) \quad \text{the limit as } w \rightarrow w_0 \text{ with } w \in V_k \text{ of } |g_n(w) - g_m(w)| \text{ is less than } \varepsilon.$$

$$(3) \quad |\lambda_k^n - \lambda_k^m| < 2\varepsilon.$$

$$\text{and (4) } \|f_n - f_m\| < 3\varepsilon.$$

Thus if  $(g_n)_{n=1}^\infty$  is Cauchy, so are  $(f_n)_{n=1}^\infty$  and  $(\lambda_k^n)_{n=1}^\infty$ . If these last two sequences have limits  $f$  and  $\lambda_k$  respectively, (in  $C(\Omega)$  and  $\mathbb{R}$ ), then  $g = f + \sum_{k=0}^\infty \lambda_k \chi_k$  is the limit in  $X$  of  $(g_n)_{n=1}^\infty$ .

With the point-wise partial ordering  $X$  is normal. We claim that  $L(X, Y)$  is not positively generated.

Let  $T : f + \sum_{k=0}^\infty \lambda_k \chi_k \mapsto f$ . We know that  $T$  is well-defined, and it is clearly linear. Furthermore  $T$  is bounded.

To show this we shall show that

$$\sup \{ |f(w)| : w \in V_k \cup \{w_0\} \} \leq 3 \sup \{ |f(w) + \lambda_k| : w \in V_k \} \vee 3 |f(w_0)| ,$$

whatever  $\lambda_k$  may be. It will follow that  $\|T\| \leq 3$ .

Now we have

$$\sup \{ |f(w) + \lambda_k| : w \in V_k \} \vee |f(w_0)| \geq (\tfrac{1}{2} v_k(f)) \vee |f(w_0)| ,$$

where  $v_k(f) = \sup \{ f(w) : w \in V_k \} - \inf \{ f(w) : w \in V_k \}$ .

On the other hand, we also have

$$\sup \{ |f(w)| : w \in V_k \} \vee |f(w_0)| \leq (|f(w_0)| + v_k(f)) \vee |f(w_0)| .$$

If  $\tfrac{1}{2} v_k(f) \leq |f(w_0)|$  then

$$3 \sup \{ |f(w) + \lambda_k| : w \in V_k \} \vee 3 |f(w_0)| \geq 3 |f(w_0)| \geq \sup \{ |f(w)| : w \in V_k \cup \{w_0\} \} ;$$

whilst if  $\tfrac{1}{2} v_k(f) \geq |f(w_0)|$ , then

$$3 \sup \{ |f(w) + \lambda_k| : w \in V_k \} \vee 3 |f(w_0)| \geq 3/2 v_k(f) \geq \sup \{ |f(w)| : w \in V_k \cup \{w_0\} \} .$$

In either case the result is proved.

If  $L(X, Y)$  were positively generated, there would be  $S \in L(X, Y)$  with  $S \geq T, 0$ . We consider  $S1$ . For each  $n$ , we have

$$S1 \geq S \left[ \sum_{k=0}^n \chi_k \right] = \sum_{k=0}^n [S\chi_k] . \quad (1).$$

If  $w \in V_k$ , let  $f_{k,w}$  be any continuous function on  $\Omega$  with  $f_{k,w}(w) = 1$ ,  $f_{k,w}|_{\complement V_k} \equiv 0$ , and  $0 \leq f_{k,w} \leq 1$ . (Such a function exists by Urysohn's lemma.) We have  $\chi_k \geq f_{k,w} \geq 0$ , and as  $S \geq T, 0$  we find that

$$\begin{aligned} S\chi_k &\geq Sf_{k,w} \\ &\geq Tf_{k,w} \\ &= f_{k,w}. \end{aligned}$$

In particular  $S\chi_k(w) \geq 1$  whenever  $w \in V_k$ . As  $S\chi_k$  is continuous,  $S\chi_k(w_0) \geq 1$ . From (1), we then have  $\{S1\}(w_0) \geq n+1$ , for all  $n$ . Clearly this is impossible, and the result is proven.

### 2.3. Special spaces.

In this section we take a brief look at spaces of linear operators between certain of our special spaces. Most of what is known is due to Ellis [3]. We present here a brief exposition of his results, with a few minor alterations.

Because all of our special spaces are positively generated, there can be few cases in which  $L(X, Y)$  is one of these. The only positive result known is Theorem 2.2.10. The converse of this result is also known.

**THEOREM 2.3.1. (ELLIS).** If  $L(X, Y)$  is order unit normed, then  $X$  is base-normed and  $Y$  is order unit normed.

Proof. Let  $B$  be the closed set  $\{x \in X_+ : \|x\| = 1\}$ . If  $f \in X^*$ ,  $y \in Y$  let  $(f \otimes y)(x) = f(x)y$ , so that  $\|f \otimes y\| = \|f\| \|y\|$ .

Let  $E$  be the order unit defining the norm in  $L(X, Y)$ , so that

$$\begin{aligned} \|f\| \|y\| &= \inf \{ \lambda : -\lambda E \leq f \otimes y \leq \lambda E \} \\ &= \inf \{ \lambda : -\lambda E(x) \leq f(x)y \leq \lambda E(x), \forall x \in B \}. \end{aligned}$$

If  $b \in B$ , there is  $f \in X^*$  with  $\|f\| = f(b) = 1$ . If  $y \in Y$  and  $\|y\| = 1$ , then we obtain:

$$\inf \{ \lambda : -\lambda E(x) \leq f(x)y \leq \lambda E(x), \forall x \in B \} = 1,$$

thus as  $Y_+$  is closed,

$$-E(b) \leq y \leq E(b),$$

so  $Y_1 \subset [-E(b), E(b)]$ . If  $b' \in B$ , then  $\|E(b')\| \leq 1$  so  $E(b') \leq E(b)$ . By symmetry  $E(b) = E(b') = e \in Y_+$ . Thus  $e$  is an order unit for  $Y$ .

If  $y \in [-e, e]$ , and  $f \in X^*$  with  $\|f\| = 1$ , then

$$-e \leq f(x)y \leq e$$

for all  $x \in B$ , so

$$\begin{aligned} \|y\| &= \|f \otimes y\| \\ &= \inf \{ \lambda : -\lambda e \leq f(x)y \leq \lambda e, \forall x \in B \} \\ &\leq 1. \end{aligned}$$

I.e.  $Y_1 = [-e, e]$ , so  $Y$  has the order unit norm defined by  $e$ .

Now Proposition 2.1.2 tells us that  $X$  is  $(1 + \varepsilon)$ -generated for all  $\varepsilon > 0$ , and the second part of the proof of Theorem 2.2.3 tells us that the norm is additive on  $X_+$ . Hence  $X$  is base-normed.

Although Corollary 2.2.14 tells us that if  $X$  is base-normed and  $Y$  is a.o.u.-normed, then  $L(X, Y)$  is not necessarily a.o.u.-normed, we might ask if the converse is true. That it is we see from:

THEOREM 2.3.2. If  $L(X, Y)$  is a.o.u.-normed, then  $X$  is base-normed and  $Y$  is a.o.u.-normed.

Proof. Let  $x \in X_+$ , with  $\|x\| = 1$ , and  $f_1, f_2 \in Y_+^*$ .  $x \otimes f_1$  can

be interpreted as a positive linear functional on  $L(X, Y)$ . As  $L(X, Y)^*$  is base-normed,

$$\begin{aligned}\|f_1\| + \|f_2\| &= \|x \otimes f_1\| + \|x \otimes f_2\| \\ &= \|x \otimes (f_1 + f_2)\| \\ &= \|f_1 + f_2\|.\end{aligned}$$

As  $L(X, Y)$  is 1-normal, the proof of Theorem 2.1.1 tells us that so also is  $Y$ . We thence see that  $Y^*$  is 1-generated, so that  $Y^*$  is base-normed and  $Y$  a.o.u.-normed.

Similarly the norm on  $X_+$  is additive, and Proposition 2.1.3 tells us that  $X$  is  $(1 + \epsilon)$ -generated for all  $\epsilon > 0$ . Hence  $X$  is base-normed.

It is also known that if  $X$  is order unit normed and  $Y$  base-normed, then  $L(X, Y)$  is not necessarily positively generated, and even if it is, it is not necessarily base-normed. In fact unless the unit ball in  $Y$  is order-bounded we can always find an order unit normed  $X$  such that  $L(X, Y)$  is not positively generated.

**PROPOSITION 2.3.3.** If  $L(X, Y)$  is positively generated whenever  $X$  is order unit normed, then  $Y_1$  is order-bounded.

Proof. Let  $E = \ell_1(Y_1)$ , with the natural partial ordering, and let  $X = E \times \mathbb{R}$ . Define  $(x, t) \geq 0$  if and only if  $\exists y \in E$ , such that  $x + ty \geq 0$  and  $t \geq 0$ . Then clearly  $X_+ \cap E = E_+$ , and also  $(0, 1)$  is an order unit for  $X$ . Let  $\|\cdot\|_0$  be the order unit norm on  $X$ . On the subspace  $E$  the norm satisfies

$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq 2 \|\cdot\|_0.$$

Let  $e_y$  denote the element of  $E$  which is 1 at  $y \in Y_1$ , and 0 for all other co-ordinates. Define  $T \in L(X, Y)$  by

$$T(e_\gamma, 0) = \gamma$$

$$T(0, 1) \in Y_1 \quad (\text{arbitrary})$$

and extend to the whole of  $X$  by linearity.  $T$  is bounded, for if  $\|(x, t)\|_0 \leq 1$ , then  $(1 + \varepsilon)(0, 1) \geq (x, t) \geq (1 + \varepsilon)(0, -1)$ , for all  $\varepsilon > 0$ . Hence  $1 \geq t \geq -1$ , so that  $\|(0, t)\|_0 \leq 1$ . Then also,

$$\begin{aligned} \|(x, 0)\|_0 &\leq \|(x, t)\|_0 + \|(0, t)\|_0 \\ &\leq 2, \end{aligned}$$

so that  $\|x\|_1 \leq 4$ . Hence

$$\|T(x, t)\| \leq \|T(x, 0)\| + \|T(0, t)\| \leq 5.$$

If  $L(X, Y)$  were positively generated, there would be  $S \in L(X, Y)$  with  $S \geq T, 0$ . We claim that  $S(0, 1)$  would be an upper bound for the unit ball of  $Y$ . In fact if  $x \in E$  with  $\|x\| \leq 1$ , then  $(0, 1) \geq (x, 0)$ . Putting  $x = e_\gamma$  we see that

$$\begin{aligned} S(0, 1) &\geq S(e_\gamma, 0) \\ &\geq T(e_\gamma, 0) \\ &= \gamma, \end{aligned}$$

for every  $\gamma \in Y_1$  as claimed.

In this case we can tell exactly when  $L(X, Y)$  can be base-normed.

**THEOREM 2.3.4. (ELLIS).** If  $L(X, Y)$  is base-normed, then  $X$  is a.o.u.-normed and  $Y$  is base-normed. If further the norm-defining base in  $L(X, Y)$  is closed for the strong operator topology, then  $X$  is order unit normed.

Proof. Proposition 2.2.1 tells us that  $X_+$  is 1-normal, and  $Y$  is  $(1 + \varepsilon)$ -generated for all  $\varepsilon > 0$ .

If  $f_1, f_2 \in X_+^*$ , and  $y \in Y_+$  with  $\|y\| = 1$ , then  $f_1 \otimes y$  have interpretations as positive elements of  $L(X, Y)$ . As  $L(X, Y)$

is base-normed, the norm is additive on the positive cone.

Hence

$$\begin{aligned}\|f_1\| + \|f_2\| &= \|f_1 \otimes y\| + \|f_2 \otimes y\| \\ &= \|(f_1 + f_2) \otimes y\| \\ &= \|f_1 + f_2\|.\end{aligned}$$

Combining this with  $X^*$  being  $(1+\varepsilon)$ -generated for all  $\varepsilon > 0$ , we see that  $X^*$  is base-normed, so  $X$  is a.o.u.-normed by Theorem 1.1.12.

A similar argument shows that the norm is additive on  $Y_+$ , so that  $Y$  is base-normed as claimed.

The last remark follows from the fact that the base  $B = \{f \in X_+^* : \|f\| = 1\}$  is weak\*-compact if the corresponding base  $\mathcal{B}$  of  $L(X, Y)$  is strong operator closed. In fact if  $(f_\alpha)$  is a net in  $B$ , then it certainly has a subnet  $(f_{\alpha'})$  convergent to some point of  $X_+^*$ ,  $f$ , say. Let  $b \in Y_+$ , with  $\|b\| = 1$ , and then  $(f_{\alpha'} \otimes b)$  is a net in  $\mathcal{B}$  converging to  $f \otimes b$  in the strong operator topology. Hence  $f \otimes b \in \mathcal{B}$ , so that  $f \in B$  and hence  $B$  is weak\*-compact. Now Theorem 1.1.8 tells us that  $X$  is order unit normed.

#### 2.4. Order structure.

In this section we shall restrict ourselves to the case when  $X_+$  and  $Y_+$  are normal and generating.

PROPOSITION 2.4.1. Let  $X, Y$  be partially ordered Banach spaces with closed, normal and generating cones. Then the following are equivalent:

- (a)  $L(X, Y)$  is conditionally complete.
- (b)  $X$  has the R.D.P. and  $Y$  is a complete vector lattice.

Proof. (b)  $\Rightarrow$  (a) : Suppose  $\{T_\alpha : \alpha \in A\}$  is a subset of  $L(X, Y)$  and that each  $T_\alpha \leq T_0 \in L(X, Y)$ . Let  $x \in X_+$ , and  $x = \sum_{k=1}^n x_k$  with  $x_k \in X_+$ ; then we have

$$\sum_{k=1}^n T_{\alpha_k} x_k \leq \sum_{k=1}^n T_0 x_k = T_0 x,$$

for any  $\alpha_k \in A$ . Thus the supremum of the set

$$\left\{ \sum_{k=1}^n T_{\alpha_k} x_k : x = \sum_{k=1}^n x_k, x_k \in X_+, \alpha_k \in A, n=1, 2, 3, \dots \right\}.$$

exists in  $Y$ . Denote this by  $Sx$ . As  $X$  has the R.D.P. it follows

by a standard argument that the map  $S : x \mapsto Sx$  is additive on the positive cone of  $X$ . Thus  $S$  can be extended to a linear operator from  $X$  to  $Y$ . It is clear that  $S \geq T_\alpha$ , and that if  $U$  is any linear operator from  $X$  to  $Y$  with  $U \geq T_\alpha$ , then  $U \geq S$ .

It remains to show that  $S \in L(X, Y)$ .

Now  $(T_0 - T_\alpha)x \geq (T_0 - S)x \geq 0$  if  $x \geq 0$ . As  $X$  is  $C$ -generating for some  $C$ , we can write each  $x \in X$ , with  $\|x\| \leq 1$ , as  $x^+ - x^-$  with  $x^+, x^- \in X_+$  and  $\|x^+\| + \|x^-\| \leq C$ . We then have:

$$-x^- \leq x \leq x^+$$

so that

$$-(T_0 - S)x^- \leq (T_0 - S)x \leq (T_0 - S)x^+.$$

But also we have,

$$(T_0 - S)x^\pm \leq (T_0 - T_\alpha)x^\pm,$$

so that

$$-(T_0 - T_\alpha)x^- \leq (T_0 - S)x \leq (T_0 - T_\alpha)x^+.$$

As  $Y$  is  $D$ -normal for some  $D$ , we see that

$$\begin{aligned} \|(T_0 - S)x\| &\leq D \max \{ \|(T_0 - T_\alpha)x^-\|, \|(T_0 - T_\alpha)x^+\| \} \\ &\leq CD \|T_0 - T_\alpha\|. \end{aligned}$$

Hence  $T_0 - S \in L(X, Y)$ . But  $T_0 \in L(X, Y)$  by assumption, so  $S \in L(X, Y)$ .

(a)  $\Rightarrow$  (b) : As  $Y$  is assumed to be positively generated, it will suffice to show that  $Y$  is conditionally complete. Let  $g$  be a non-zero bounded positive linear functional on  $X$ , with

$g(x_0) = 1$ , for  $x_0 \in X_+$ , and  $\|x_0\| = 1$ . Let  $(s_i)_{i \in I}$  be a family in  $Y$ , bounded above by  $m$ . Define  $S_i : x \mapsto g(x)s_i$ , with  $M$  defined similarly. If  $x \geq 0$ , then  $(M - S_i)x = (m - s_i)g(x) \geq 0$ , so that  $M \geq S_i$ . Let  $T$  be the supremum of  $(S_i)_{i \in I}$  in  $L(X, Y)$ , which exists by assumption. As  $T \geq S_i$ , we have  $Tx_0 \geq S_ix_0 = s_i$ . On the other hand, if  $t \geq s_i$ , defining  $T' : x \mapsto g(x)t$ , we have  $T' \geq S_i$ . Thus  $t = T'x_0 \geq Tx_0$ . I.e.  $Tx_0$  is the supremum of  $(s_i)_{i \in I}$ .

To prove that  $X$  has the R.D.P., note firstly that  $L(X, Y)$  has the R.D.P. Let  $f$  be a non-zero, bounded positive linear functional on  $Y$ , with  $f(y_0) = \|y_0\| = 1$  for  $y_0 \in Y_+$ . If  $g, h \geq m, n$  with all these elements of  $X^*$ , define  $G : x \mapsto g(x)y_0$ , etc. Then  $G, H \geq M, N$  and all belong to  $L(X, Y)$ . Thus  $\exists L \in L(X, Y)$  with  $G, H \geq L \geq M, N$ . Composing with  $f$  we see that

$$g, h \geq f \circ L \geq m, n$$

and clearly  $f \circ L \in X^*$ . It follows that  $X^*$  has the R.D.P., and hence by Theorem 1.3.1 so has  $X$ .

If we limit  $Y$  so that  $L(X, Y)$  is positively generated, then we can obtain the following:

COROLLARY 2.4.2. Let  $\Omega$  be a Stonian space, and  $X$  a partially ordered Banach space with a closed, normal and generating cone. Then the following are equivalent:

- (a)  $L(X, C(\Omega))$  is a complete vector lattice.
- (b)  $X$  has the R.D.P.

It would be of interest to know when, for example,  $L(X, Y)$  has the R.D.P. It is easily seen that both  $X$  and  $Y$

will have to have the R.D.P. However in general there appears to be no obvious means of proving the implication in the reverse direction. In the case when  $X$  is finite dimensional, the following results are easily seen to hold, but we can prove no more general result.

PROPOSITION 2.4.3. Let  $X, Y$  be partially ordered Banach spaces with closed, normal and generating cones, and suppose that  $X$  is a finite dimensional lattice. Then we have the following:

(a)  $L(X, Y)$  is a lattice iff  $Y$  is a lattice.

(b)  $L(X, Y)$  has the R.D.P. iff  $Y$  has the R.D.P.

## 2.5. Projective tensor products.

We follow the notation of Schatten [1]. If  $X$  and  $Y$  are Banach spaces,  $X \odot Y$  will denote their algebraic tensor product. We shall consider only the greatest cross-norm  $\gamma$ . If  $X \odot Y$  is represented in the usual manner as equivalence classes of expressions  $\sum_{i=1}^n x_i \otimes y_i$ , then

$$\left\| \sum_{i=1}^n x_i \otimes y_i \right\|_{\gamma} = \inf \left\{ \sum_{j=1}^m \|x'_j\| \|y'_j\| : \sum_{j=1}^m x'_j \otimes y'_j \text{ is equivalent to } \sum_{i=1}^n x_i \otimes y_i \right\}.$$

with this norm we write the space as  $X \odot_{\gamma} Y$ , and  $X \otimes_{\gamma} Y$  denotes its completion.

It is known that  $(X \otimes_{\gamma} Y)^*$  is linearly isometric to  $L(X, Y^*)$ . If  $T \in L(X, Y^*)$  then the associated linear functional on  $X \otimes_{\gamma} Y$  maps  $\sum_{i=1}^n x_i \otimes y_i$  to  $\sum_{i=1}^n (Tx_i) y_i$ . It is easily seen that  $T$  is positive if and only if it is positive on all such finite expressions for  $x_i \in X_+$ ,  $y_i \in Y_+$ . Accordingly we define the positive wedge in  $X \otimes_{\gamma} Y$  to be the closure of the set

$$\left\{ \sum_1^n x_i \otimes y_i : x_i \in X_+, y_i \in Y_+ \right\}.$$

With this ordering  $L(X, Y^*)$  is isometrically order-isomorphic to  $(X \otimes_y Y)^*$ . We are thus able to deduce properties of  $X \otimes_y Y$  from our results for spaces of bounded linear operators, and the general duality results.

PROPOSITION 2.5.1.  $X \otimes_y Y$  is positively generated if and only if both  $X$  and  $Y$  are positively generated.

Proof.

$X \otimes_y Y$  is positively generated  $\iff (X \otimes_y Y)_+^*$  is normal.

$\iff L(X, Y^*)_+$  is normal.

$\iff X_+$  is generating and  $Y_+^*$  normal.

$\iff X_+$  and  $Y_+$  are generating.

In general, of course,  $(X \otimes_y Y)_+$  is not normal, even if both  $X_+$  and  $Y_+$  are both normal. However translating Proposition 2.2.8 we obtain:

PROPOSITION 2.5.2. If  $X$  is finite dimensional and  $Y_+$  is normal, then  $(X \otimes_y Y)_+$  is normal.

We can also obtain:

PROPOSITION 2.5.3. If  $X_+$  is normal and  $Y$  is base-normed with the R.D.P., then  $(X \otimes_y Y)_+$  is normal.

Proof. By Theorem 1.1.10  $Y^*$  is order unit normed, and by Theorem 1.3.1 it is a complete vector lattice. Now Theorem 2.2.4 translates to give this result.

The final case in which we can conclude that the cone in the tensor product is normal is the following, which follows from

Theorem 2.2.11 and Proposition 1.2.4.

PROPOSITION 2.5.4. If there exist constants  $\alpha$  and  $\beta$  such that  $X$  is  $\alpha$ -additive and  $Y$  is  $\beta$ -additive, then  $(X \otimes_{\gamma} Y)_+$  is normal.

We can also obtain some results involving the special types of spaces that we have considered. The first of these is closely related to the last result. It follows from a retranslation of Theorems 2.2.10 and 2.3.1.

PROPOSITION 2.5.5.  $X \otimes_{\gamma} Y$  is base-normed if and only if both  $X$  and  $Y$  are base-normed.

The other results that we have are not so complete, consisting of implications in one direction only. Both of them are deduced from Theorem 2.3.4, together with the symmetry of the tensor product.

PROPOSITION 2.5.6. If  $(X \otimes_{\gamma} Y)$  is a.o.u.-normed then so are both  $X$  and  $Y$ .

PROPOSITION 2.5.7. If  $(X \otimes_{\gamma} Y)$  is order unit normed, then so are both  $X$  and  $Y$ .

Proof. The base of  $L(X, Y^*)$  is compact for the weak\*-topology as the dual of  $X \otimes_{\gamma} Y$ , and hence certainly closed. The base is then certainly closed for the (stronger) strong operator topology of  $L(X, Y^*)$ , so the last part of Theorem 2.3.4 may be invoked.

If  $(X \otimes_{\gamma} Y)_+$  is normal, and both  $X_+$  and  $Y_+$  are normal and generating, then we can say something about the order properties of the tensor product. We have in fact:

PROPOSITION 2.5.8. If  $(X \otimes_Y Y)_+$  is normal and generating, then the following are equivalent:

(a)  $X \otimes_Y Y$  has the R.D.P.

(b) Both  $X$  and  $Y$  have the R.D.P.

Proof. Both  $X_+$  and  $Y_+$  must be normal and generating for  $(X \otimes_Y Y)_+$  to be so. Then Theorem 1.3.1 tells us that (a) is equivalent to  $L(X, Y^*)$  being a complete vector lattice. By Proposition 2.4.1 this is equivalent to  $X$  having the R.D.P. and  $Y^*$  being a complete vector lattice. Now again invoking Theorem 1.3.1 shows us that this is equivalent to both  $X$  and  $Y$  having the R.D.P.

As a special case we have:

PROPOSITION 2.5.9. Let  $Y$  be base-normed and have the R.D.P., and  $X_+$  normal and generating. Then  $X$  has the R.D.P. if and only if  $X \otimes_Y Y$  has the R.D.P.

If  $X$  and  $Y$  are Banach spaces, and  $\sum_{i=1}^n f_i \otimes y_i \in X^* \odot Y$ , then the formula

$$x \mapsto \sum_{i=1}^n f_i(x) y_i$$

defines an element of  $L(X, Y)$ . This correspondence is a continuous map from  $X^* \odot Y$  into  $L(X, Y)$ . As  $L(X, Y)$  is complete, this can be extended to a map of  $X^* \otimes_Y Y$  into  $L(X, Y)$ . The elements of the image of  $X^* \otimes_Y Y$  under this map are termed the nuclear operators from  $X$  to  $Y$ . Let  $N(X, Y)$  denote this subspace of  $L(X, Y)$ . We can invoke Theorem 2.1.1 to give us immediately:

PROPOSITION 2.5.10.  $N(X, Y)_+$  is normal if and only if  $X$  is positively generated and  $Y_+$  is normal.

We can also obtain:

PROPOSITION 2.5.11.  $N(X, Y)$  is positively generated if and only if  $X_+$  is normal and  $Y$  is positively generated.

Proof. If  $\sum_{i=1}^n f_i \otimes y_i \in (X^* \otimes_Y Y)_+$ , then we may assume that each  $f_i$  and  $y_i$  are positive. If  $x \in X_+$ , then  $\sum f_i(x) y_i$  is positive. Now if  $v \in (X^* \otimes_Y Y)_+$ , then  $v$  is a limit of such terms, so that the associated nuclear operator is positive, (as  $Y_+$  is closed).

Now let  $T \in N(X, Y)$ , so that  $T$  arises from an element  $v$  of  $X^* \otimes_Y Y$ . As both  $X^*$  and  $Y$  are positively generated, so is  $X^* \otimes_Y Y$ , and hence  $v = v^+ - v^-$ , with  $v^+, v^- \in (X^* \otimes_Y Y)_+$ . By the remarks in the last paragraph, the nuclear operators associated with  $v^+$ , and  $v^-$  are positive and give the desired decomposition.

The converse follows as in the proof of Proposition 2.2.1.

Ordered tensor products have been studied by Ellis [3], and Peressini and Sherbert [1]. Tensor products of compact convex sets have been dealt with by Davies and Vincent-Smith [1], Lazar [3], and Namioka and Phelps [1].

## Chapter III

Spaces of compact linear operators between partially ordered  
Banach spaces.

We devote ourselves in this chapter to a study of spaces of compact linear operators between partially ordered Banach spaces. In order to obtain results it is necessary to make some drastic limitation on the spaces involved. In view of Lindenstrauss's results, [1], it would seem desirable that the dual of the range space be an  $L^1(\mu)$  space. We therefore take the range to be a simplex space in all that follows. The first section will deal with conditions for the space to have a normal cone, be positively generated, or to have some special form. In the second section we look at the order structure of the spaces involved.

3.1. Normality, positive generation, and special spaces.

$K(X, Y)$  will denote the Banach space of all compact linear operators from  $X$  to  $Y$ , with the supremum norm, and ordering inherited from  $L(X, Y)$ . As a simplex space is 1-normal, we obtain immediately from Corollary 2.1.2:

PROPOSITION 3.1.1. Let  $Y$  be a simplex space.  $X$  is  $(C + \epsilon)$ -generating for all  $\epsilon > 0$  if and only if  $K(X, Y)_+$  is  $C$ -normal.

From Corollary 2.1.8 we similarly obtain:

PROPOSITION 3.1.2. Let  $Y$  be a simplex space, and  $\alpha \geq 1$ . Then the following are equivalent:

- (a) If  $x \in X$ ,  $\|x\| < 1$ , there is  $y \geq x$ ,  $-x$  with  $\|y\| < \alpha$ .  
 (b) If  $S, T \in K(X, Y)$  with  $-S \leq T \leq S$ , then  $\|T\| \leq \alpha \|S\|$ .

We wish to know when  $K(X, Y)$  is positively generated. As long as  $Y$  is a simplex space we can obtain a precise result. However, even in this situation, we know nothing of the constants involved.

**THEOREM 3.1.3.** Let  $Y$  be a simplex space, and  $X$  a partially ordered Banach space with a closed cone.  $X_+$  is normal if and only if  $K(X, Y)$  is positively generated.

Proof. By Proposition 1.1.16, it will suffice to prove that  $A_0(K, X^*)$  is positively generated, where  $K$  is the simplex  $\{f \in Y_+^* : \|f\| \leq 1\}$ , and  $X^*$  is positively generated. ( $A_0(K, E)$  denoting the space of all continuous affine functions from  $K$  into the Banach space  $E$ , which vanish at zero, having the supremum norm, and natural partial order.)

If  $\pi \in A_0(K, X^*)$ , let

$$\begin{aligned}\bar{\Phi}(k) &= \{x \in X^* : x \geq \pi(k), 0\} & \text{if } k \neq 0. \\ &= \{0\} & \text{if } k = 0.\end{aligned}$$

Since  $X^*$  is positively generated,  $\bar{\Phi}(k) \neq \emptyset$ . It is clear that  $\bar{\Phi}(k)$  is closed, and convex. We show that  $\bar{\Phi}$  is lower semi-continuous and affine, so that Theorem 1.3.7 will ensure that there is an affine selection  $\varphi$  of  $\bar{\Phi}$ . It is clear that this  $\varphi$  satisfies  $\varphi \geq \pi, 0$ , thus completing the proof.

Suppose  $x \in \bar{\Phi}(k), x' \in \bar{\Phi}(k')$ , then  $x \geq \pi(k), 0$  and  $x' \geq \pi(k'), 0$ . It will immediately follow that

$$\lambda x + (1 - \lambda)x' \geq \pi(\lambda k + (1 - \lambda)k'), 0.$$

If  $\lambda k + (1 - \lambda)k' \neq 0$ , this shows that

$$\lambda \bar{\Phi}(k) + (1-\lambda) \bar{\Phi}(k') \subset \bar{\Phi}(\lambda k + (1-\lambda)k').$$

On the other hand if  $\lambda k + (1-\lambda)k' = 0$ , then  $k = k' = 0$ , (as 0 is an extreme point of  $K$ ) so that the inclusion is trivial. We have thus proved that  $\bar{\Phi}$  is affine.

To prove lower semi-continuity, it will suffice to prove that if  $C$  is a closed subset of  $X^*$ , then the set

$$M = \{k \in K : \bar{\Phi}(k) \subset C\},$$

is a closed subset of  $K$ . Suppose  $0 \neq k_0 \in \bar{M}$ , and that  $(k_\gamma)$  is a net in  $M$  converging to  $k_0$ . Let  $x \in \bar{\Phi}(k_0)$ , so that  $x \gg \pi(k_0), 0$ . Choose  $\gamma$  so that  $\|\pi(k_\gamma) - \pi(k_0)\| < \xi$ . For some  $D$ ,  $X^*$  is  $C$ -generated, so  $\exists y \gg \pi(k_\gamma) - \pi(k_0), 0$  with  $\|y\| < D\xi$ . It is clear that  $x+y \gg \pi(k_\gamma), 0$ , so we may suppose  $x+y \in \bar{\Phi}(k_\gamma)$ , (we may omit from the net  $(k_\gamma)$  all terms that are 0, since  $k_0 \neq 0$ , and then the result is clear.). As  $k_\gamma \in M$ ,  $x+y \in C$ . But  $\|(x+y) - x\| < D\xi$ , and  $\xi$  was arbitrary so that  $x \in \bar{C} = C$ . Hence  $\bar{\Phi}(k_0) \subset C$ , so  $k_0 \in M$ .

If  $k_0 = 0$ , and  $(k_\gamma) \subset M$  is a net converging to 0, we can obtain a net in  $\bar{\Phi}(k_\gamma)$  converging to 0. Then again we see that  $\bar{\Phi}(k_0) = \{0\} \subset C$ , so that  $0 = k_0 \in M$ . So in either case  $M$  is closed, and the result proven.

For the converse, the proof of Proposition 2.2.1 may be repeated.

If we desire to look at the general case, we can give a result that is partly analogous to Proposition 2.2.12.

PROPOSITION 3.1.4.  $K(X, Y)$  is positively generated whenever  $X$  is base-normed, if and only if each relatively compact subset of  $Y$  is order-bounded.

Proof. If  $X$  is base-normed, with base  $B$ , let  $T \in K(X, Y)$ . It follows that  $T(B)$  is a relatively compact subset of  $Y$ . Hence so also is  $\text{co}(\{0\} \cup T(B))$ . Let  $e$  be an upper bound for this latter set. Let  $f \in X_+^*$ ,  $f|_B \equiv 1$ , and define  $S \in K(X, Y)$  by  $S : x \mapsto f(x)e$ . Clearly if  $b \in B$ ,  $Sb \geq Tb$ ,  $0$ . But if  $x \in X_+$ , then  $x = \lambda b$  for some  $\lambda \geq 0$ , and  $b \in B$ . Thus we see that  $S \geq T$ ,  $0$ . It is clear that  $S$  is compact, so  $K(X, Y)$  is positively generated as claimed.

To prove the converse, suppose  $L$  is a relatively compact subset of  $Y$ , and let  $N$  be the compact convex set  $\overline{\text{co}}(L \cup -L)$ . An upper bound for  $N$  will also be an upper bound for  $L$ . Let  $M$  be the direct sum of the affine span of  $N$ , and the reals. The set  $N \times \{1\}$  forms the base of a cone in this space. If the space is given the corresponding base-norm, it will be complete and the positive cone closed. Define  $T \in K(M, Y)$  by  $T(n, 1) = n$  if  $n \in N$ , and extend by linearity. If  $S \in K(M, Y)$  with  $S \geq T$ ,  $0$ , the argument used in Proposition 2.2.12 now shows that there is an upper bound in  $Y$  for  $N$ , and hence for  $L$ .

Exactly what spaces have this property, that relatively compact sets are order-bounded, we do not know. Combining Proposition 3.1.4 with Theorem 3.1.3 shows us that all simplex spaces have this property. If we knew that for all spaces with normal, generating cones,  $X^*$  had this property if and only if  $X$  were equivalent to a base-normed space, then we could produce a result analogous to Corollary 2.2.15. In particular this would be true if the spaces involved (for  $X_+$  normal and generating) were precisely the spaces that are equivalent to an a.o.u.-normed space. Although this conjecture is tempting,

we must admit that we have no real justification for making it.

The proofs of Theorems 2.2.10 and 2.3.1 are equally valid here, so we can state:

THEOREM 3.1.5. Let  $X, Y$  be arbitrary partially ordered Banach spaces with closed cones. Then  $K(X, Y)$  is order unit normed if and only if  $X$  is base-normed and  $Y$  is order unit normed.

Unlike the case when we were dealing with bounded operators, we can extend this result.

THEOREM 3.1.6. Let  $X$  be a partially ordered Banach space with a closed cone, and  $Y$  a simplex space. Then  $K(X, Y)$  is a.o.u.-normed if and only if  $X$  is base-normed.

Proof. By Proposition 3.1.1,  $K(X, Y)_+$  is 1-normal; so to prove that  $K(X, Y)$  is a.o.u.-normed it will certainly suffice to prove that the closed unit ball of  $K(X, Y)$  is upward directed. Again we use Proposition 1.1.16 to identify  $K(X, Y)$  with  $A_0(K, X^*)$  where  $K = \{f \in Y^* : f \geq 0, \|f\| \leq 1\}$  with the weak\*-topology.

Let  $\varphi, \psi \in A_0(K, X^*)$ ;  $\|\varphi\|, \|\psi\| \leq 1$ . Theorem 1.1.10 tells us that  $X^*$  is order unit normed, so can be identified with  $A(M)$  for some compact convex set  $M$ . Let  $p$  be the real-valued function defined on  $K$  by  $p(k) = 1$  (if  $k \neq 0$ ),  $p(0) = 0$ . It is readily verified that  $p$  is lower semi-continuous and concave. Now define  $f : K \rightarrow \mathbb{R}$  as follows:

$$f(k) = \sup \{ [\varphi(k)](m) : m \in M \}.$$

Clearly  $-1 \leq f(k) \leq 1$ , and  $f(0) = 0$ .

It is also true that  $f$  is convex, since

$$\begin{aligned} f(k+k') &= \sup \{ [\varphi(k+k')] (m) : m \in M \} \\ &= \sup \{ [\varphi(k)] (m) + [\varphi(k')] (m) : m \in M \} \\ &\leq \sup \{ [\varphi(k)] (m) : m \in M \} + \\ &\quad \sup \{ [\varphi(k')] (m) : m \in M \} \\ &= f(k) + f(k'). \end{aligned}$$

We can also prove that  $f$  is lower semi-continuous.

Consider the set  $\{k : f(k) < \alpha\}$ , we shall see that this is open. If  $f(k_0) = \alpha - 2\xi$ , then there is a weak\*-neighbourhood  $U$  of  $k_0$  such that  $\|\varphi(k) - \varphi(k_0)\| < \xi$  whenever  $k \in U$ , (as  $\varphi$  is continuous). We then have:

$$\begin{aligned} f(k) &= \sup \{ [\varphi(k)] (m) : m \in M \} \\ &\leq \sup \{ [\varphi(k_0)] (m) : m \in M \} + \|\varphi(k) - \varphi(k_0)\| \\ &< f(k_0) + \xi \\ &\leq \alpha - \xi. \end{aligned}$$

It follows that  $\{k : f(k) < \alpha\}$  is open, and  $f$  is upper semi-continuous. Define  $g$  similarly, with  $\psi$  replacing  $\varphi$ , and let  $h = f \vee g$  (point-wise supremum). Clearly  $h$  is upper semi-continuous and convex. By Theorem 1.3.6, we see that there exists a continuous affine function  $q$  on  $K$ , satisfying  $h \leq q \leq p$ . Now define  $\pi : K \rightarrow X^*$ , by  $\pi(k) = q(k)1_M$ . We certainly have  $\|\pi\| \leq 1$ , and  $\pi(k) \geq \varphi(k), \psi(k)$  for all  $k \in K$ . Also  $\pi(0) = 0$ , since  $p(0) = h(0) = 0$ . Thus the implication in one direction is proved.

To prove the converse, note that Proposition 3.1.1 tells us that  $X$  is  $(1 + \xi)$ -generated for all  $\xi > 0$ . It will suffice to prove that the norm on  $X_+$  is additive. But if  $x_1, x_2 \in X$ , and  $f \in Y_+^*$ , with  $\|f\| = 1$ , then  $x_i \otimes f$  are bounded, positive linear functionals on  $K(X, Y)$ , of norm  $\|x_i\|$ . As  $K(X, Y)$  is a.o.u.-normed,

$K(X, Y)^*$  is base-normed by Theorem 1.1.12. It follows that the norm is additive on the positive cone of  $K(X, Y)^*$ , so we have:

$$\begin{aligned} \|x_1\| + \|x_2\| &= \|x_1 \otimes f\| + \|x_2 \otimes f\| \\ &= \|(x_1 + x_2) \otimes f\| \\ &= \|x_1 + x_2\|, \end{aligned}$$

so the proof is complete.

### 3.2. Order properties.

We consider here the order properties of  $K(X, Y)$  when  $X$  has a closed, normal and generating cone, and  $Y$  is a simplex space. We have firstly:

THEOREM 3.2.1. Let  $X$  be a partially ordered Banach space with a closed, normal and generating cone, and  $Y$  a simplex space. Then  $K(X, Y)$  has the R.D.P. if and only if  $X$  has the R.D.P.

Proof. The proof in one direction is identical with that of Theorem 2.4.1.

We again use Proposition 1.1.16 to reduce the problem to the space  $A_0(K, X^*)$ , and use Lazar's theorem, Theorem 1.3.7.

If  $\sigma, \tau \geq \varphi, \psi$  and all belong to  $A_0(K, X^*)$ , let

$$\Pi(k) = \{x \in X^* : \sigma(k), \tau(k) \geq x \geq \varphi(k), \psi(k)\}.$$

Any continuous affine selection for  $\Pi$  will provide us with a  $\pi \in A_0(K, X^*)$  which satisfies  $\sigma, \tau \geq \pi \geq \varphi, \psi$ .

We must now verify that  $\Pi$  satisfies all the conditions needed for us to apply Theorem 1.3.7. It is easily seen that  $\Pi(k)$  is closed (as  $X_+^*$  is closed), non-empty (as  $X^*$  is a lattice), and affine. We must show that  $\Pi$  is lower semi-

continuous, i.e. if  $U$  is open in  $X^*$ , then  $\{k : \bigcap (k) \cap U \neq \emptyset\}$  is open.

We recall that  $X^*$  is a lattice, and that the lattice operations satisfy:

$$(a \vee b)(x) = \sup \{a(y) + b(z) : y, z \geq 0; y + z = x\}$$

$$(a \wedge b)(x) = \inf \{a(y) + b(z) : y, z \geq 0; y + z = x\}$$

for all  $x \geq 0$ , since  $X_+$  is normal and generating. If  $X_+$  is in fact  $C$ -generating and  $D$ -normal, then we claim that

$$\|(a \vee b) - (c \vee d)\| \leq CD(\|a - c\| + \|b - d\|).$$

For we have, on the positive part of the unit ball of  $X$ :

$$\begin{aligned} & \sup \{a(y) + b(z) : y, z \geq 0; y + z = x\} \\ & \leq \sup \{c(y) + p(y) + d(z) + q(z) : y, z \geq 0; y + z = x\}, \\ & \quad (\text{where } p \geq a - c, 0 \text{ with } \|p\| \leq D\|a - c\|; \text{ and } q \geq b - d, 0 \text{ with } \|q\| \leq D\|b - d\|). \\ & \leq \sup \{c(y) + d(z) : y, z \geq 0; y + z = x\} + \sup \{p(y) + q(z) : y, z \geq 0; y + z = x\} \\ & \leq \sup \{c(y) + d(z) : y, z \geq 0; y + z = x\} + p(x) + q(x) \\ & \leq (c \vee d)(x) + p(x) + q(x). \end{aligned}$$

By symmetry we have:

$$\begin{aligned} |(a \vee b)(x) - (c \vee d)(x)| & \leq \|p\| + \|q\| \\ & \leq D[\|a - c\| + \|b - d\|]. \end{aligned}$$

But then we see that:

$$\begin{aligned} & \|(a \vee b) - (c \vee d)\| \\ & = \sup \{|(a \vee b)(x) - (c \vee d)(x)| : \|x\| \leq 1\} \\ & \leq \sup \{|(a \vee b - c \vee d)(x^+ - x^-)| : x^+, x^- \geq 0; \|x^+\| + \|x^-\| \leq C\} \\ & \leq C \sup \{|(a \vee b - c \vee d)(x^+)| : x^+ \geq 0, \|x^+\| \leq 1\} \\ & \leq CD[\|a - c\| + \|b - d\|]. \end{aligned}$$

We obtain similarly:

$$\|(a \wedge b) - (c \wedge d)\| \leq CD[\|a - c\| + \|b - d\|].$$

We can now show that  $\Pi$  is lower semi-continuous, and thus complete the proof. Suppose  $k_0 \in \{k : \Pi(k) \cap U \neq \emptyset\}$ , then there is an  $x_0 \in U$  such that

$$\sigma(k_0), \tau(k_0) \geq x_0 \geq \varphi(k_0), \psi(k_0).$$

As  $\sigma, \tau, \varphi$  and  $\psi$  are all continuous, we can find a neighbourhood  $N(\varepsilon)$  of  $k_0$  such that  $\|\sigma(k_0) - \sigma(k)\| < \varepsilon$  whenever  $k \in N(\varepsilon)$ , and similar inequalities hold involving  $\tau, \varphi$  and  $\psi$ .

Let  $x$  be defined to be:

$$(x_0 + (\sigma(k) - \sigma(k_0)) \wedge (\tau(k) - \tau(k_0))) \vee \varphi(k) \vee \psi(k).$$

It is clear that

$$\sigma(k), \tau(k) \geq x \geq \varphi(k), \psi(k)$$

so that  $x \in \Pi(k)$ . We find a limit for  $\|x - x_0\|$ , which is equal to:

$$\begin{aligned} & \| (x_0 + (\sigma(k) - \sigma(k_0)) \wedge (\tau(k) - \tau(k_0))) \vee \varphi(k) \vee \psi(k) - x_0 \| \\ &= \| (x_0 + (\sigma(k) - \sigma(k_0)) \wedge (\tau(k) - \tau(k_0))) \vee \varphi(k) \vee \psi(k) - \\ & \quad x_0 \vee \varphi(k_0) \vee \psi(k_0) \| \\ &\leq CD(\| (x_0 + (\sigma(k) - \sigma(k_0)) \wedge (\tau(k) - \tau(k_0))) \vee \varphi(k) - x_0 \vee \varphi(k_0) \| + \varepsilon) \\ &\leq CD(CD(\| x_0 + (\sigma(k) - \sigma(k_0)) \wedge (\tau(k) - \tau(k_0)) - x_0 \| + \varepsilon) + \varepsilon) \\ &= \varepsilon (CD + C^2 D^2) + C^2 D^2 \| (\sigma(k) - \sigma(k_0)) \wedge (\tau(k) - \tau(k_0)) - 0 \wedge 0 \| \\ &\leq \varepsilon (CD + C^2 D^2) + C^2 D^2 (CD(\varepsilon + \varepsilon)) \\ &= \varepsilon (CD + C^2 D^2 + 2C^3 D^3). \end{aligned}$$

As  $x_0 \in U$ ,  $\exists \delta > 0$  such that  $\{z : \|z - x_0\| < \delta\} \subset U$ .

If we let  $\varepsilon = \delta / (CD + C^2 D^2 + 2C^3 D^3)$ , then  $x \in U$  whenever  $k \in N(\varepsilon)$ .

It follows that  $\Pi$  is lower semi-continuous as stated.

We now look at conditions for  $K(X, Y)$  to be a lattice.

The result here was stated without proof by Krengel in [1].

We first need the following lemma. Note now that if  $A_0(K)$  is a lattice, then the lattice operations are point-wise operations

on  $\partial_e K$ .

LEMMA 3.2.2. Let  $Y$  be an a.o.u.-normed lattice, ( i.e. an  $M$ -space ) and  $C$  a relatively compact subset of  $Y$ . Then  $\sup(C)$  exists in  $Y$ , and the set

$$C' = \{ \sup(A) : A \subset C \},$$

is relatively compact.

Proof. Let  $\{e_\lambda : \lambda \in \Lambda\}$  be an approximate order unit for  $Y$  which defines the norm in  $Y$ . We may, since  $Y$  is a lattice, assume that  $\|y\| \leq 1$  if and only if  $\exists \lambda \in \Lambda$  with  $e_\lambda \geq y \geq -e_\lambda$ .

Choose a sequence  $\{x_k\} \subset C$  such that for each  $n$ ,  $\exists N_n$  with

$$C \subset \bigcup_{k=1}^{N_n} S(x_k, 1/n),$$

( where  $S(a, r)$  is the open ball of centre  $a$  and radius  $r$  ).

Let  $y_n = \sup\{x_1, \dots, x_{N_n}\}$ . Given  $\varepsilon > 0$ , choose  $n$  with  $(1/n) \leq \varepsilon$ .

Suppose that  $p \geq N_n$ , and  $1 \leq q \leq p$ , then there exists  $j$  with  $1 \leq j \leq N$  and  $x_q \in S(x_j, 1/n)$ . Then, for some  $\lambda(q) \in \Lambda$ ,

$$\begin{aligned} x_q &\leq x_j + (1/n)e_{\lambda(q)} \\ &\leq y_{N_n} + (1/n)e_{\lambda(q)}. \end{aligned}$$

It follows that

$$y_{N_n} \leq y_p \leq y_{N_n} + (1/n)e_\mu$$

where  $\mu \geq \lambda(q)$  for  $1 \leq q \leq p$ . It follows that  $(y_k)$  is a Cauchy sequence in  $Y$ , so converges to some  $y \in Y$ . As  $(y_k)$  is increasing, and the positive cone in  $Y$  is closed,  $y = \sup\{y_k\}$ .

If  $x \in C$  and  $\varepsilon > 0$ , then there is  $n_0$  such that  $\|x - x_{n_0}\| \leq \varepsilon$ .

Then we have:

$$\begin{aligned} x &\leq x_{n_0} + \varepsilon e_{\lambda(\varepsilon)} \\ &\leq y_{n_0} + \varepsilon e_{\lambda(\varepsilon)} \\ &\leq y + \varepsilon e_{\lambda(\varepsilon)}, \end{aligned}$$

( for some  $\lambda(\xi) \in \Lambda$ ). I.e.  $x \leq y + \xi e_{\lambda(\xi)}$ . Letting  $\xi \rightarrow 0$ , and noting that  $Y_+$  is closed, we see that  $y \geq x$ . Hence  $y$  is an upper bound for  $C$ . On the other hand, if  $z$  is any other upper bound for  $C$ , then  $z \geq y_k$ . Hence  $z \geq \sup \{y_k\} = y$ , so that  $y = \sup(C)$ . Note here that from the proof,  $y$  is the point-wise supremum of  $C$  on  $\partial_e K$ .

To prove the second part, let  $\xi > 0$ ,  $(1/n) < \xi$  and  $\mathcal{A}$  the class of all finite subsets of  $\{1, \dots, N_n\}$ . If  $A \subset C$ , choose  $\{k(1), \dots, k(r)\} \in \mathcal{A}$  so that:

- (a)  $\forall x \in A, \exists x_{k(j)}, (1 \leq j \leq r)$ , with  $x \in S(x_{k(j)}, 1/n)$ .
- (b)  $S(x_{k(j)}, 1/n) \cap A \neq \emptyset$  for  $1 \leq j \leq r$ .

It follows that for  $1 \leq j \leq r$ ,  $\exists \lambda(j)$  with

$$x_{k(j)} \leq x + (1/n)e_{\lambda(j)}, \text{ for some } x \in A.$$

But  $A$  is relatively compact, so  $\sup(A)$  exists, and we have

$$x_{k(j)} \leq \sup(A) + (1/n)e_{\lambda(j)}.$$

We then have:

$$\sup \{x_{k(j)} : 1 \leq j \leq r\} \leq \sup(A) + (1/n)e_{\mu} \quad (1)$$

(where  $\mu \geq \lambda(j)$ , for  $1 \leq j \leq r$ ). If  $x \in A$ ,  $\exists j$  with  $1 \leq j \leq r$  and  $e_{\nu(j)}$  such that:

$$x \leq x_{k(j)} + (1/n)e_{\nu(j)} \quad (\text{by (a)}).$$

Then

$$x \leq \sup \{x_{k(j)} : 1 \leq j \leq r\} + (1/n)e_{\gamma(x)}$$

( where  $\gamma(x) \geq \nu(j)$ , for  $1 \leq j \leq r$  ). Now:

$$\sup(A) \leq \sup \{x_{k(j)} : 1 \leq j \leq r\} + (1/n)1_K, \quad (2)$$

since the supremum of  $A$  is the pointwise supremum on  $\partial_e K$ .

Combining (1) and (2), we see that

$$\|\sup(A) - \sup \{x_{k(j)} : 1 \leq j \leq r\}\| \leq 1/n < \xi.$$

Thus the finite set  $\{\sup \{x_j : j \in \alpha\} : \alpha \in \mathcal{A}\}$  is an  $\xi$ -net covering  $\{\sup(A) : A \subset C\}$ , which implies the desired result.

THEOREM 3.2.3. Let  $X$  be a partially ordered Banach space with a closed, normal and generating cone, and  $Y$  a simplex space.

Then the following are equivalent:

(a)  $K(X, Y)$  is a lattice.

(b)  $X$  has the R.D.P. and  $Y$  is a lattice.

Proof. (a) $\Rightarrow$ (b) is proved in the same manner as the same implication in Theorem 2.4.1.

(b) $\Rightarrow$ (a): Suppose  $T \in K(X, Y)$ . Let

$$S_y = \sup \{Tx : y \geq x \geq 0\} \quad (\text{if } y \geq 0),$$

which exists since  $T$  is compact,  $X_+$  is normal, and by Lemma 3.2.2.

As  $X$  has the R.D.P.  $S$  is additive on  $X_+$ , so extends to a linear operator from  $X$  to  $Y$ . It is clear that  $S$  is the supremum of  $T$  and  $0$  in  $L(X, Y)$ . We need now only show that  $S$  is compact.

But

$$SX_{1+} \subset \left\{ \sup \{Tx : y \geq x \geq 0\} : y \in X_{1+} \right\},$$

is relatively compact by Lemma 3.2.2. Hence

$$SX_1 \subset \text{co} (SX_{C+} \cup -SX_{C+})$$

is relatively compact, so that  $S$  is compact. Hence  $K(X, Y)$  is a lattice.

Even if  $X$  has the R.D.P. and  $Y$  is an order unit normed complete vector lattice,  $K(X, Y)$  need not be a complete vector lattice, since we have the following:

EXAMPLE 3.2.4. Let  $T_i \in K(\ell_1, \ell_\infty)$  be defined to be the restrictions of the natural injection to the first  $i$  co-ordinates. The family  $\{T_i\}_{i=1}^\infty$  of compact operators is bounded above by the compact linear operator  $S$  mapping  $(x_n)$  to the sequence that is constantly  $\sum_{n=1}^\infty (x_n)$ . If  $\bigvee_{i=1}^\infty T_i$  existed, we claim that

it would be the point-wise supremum. If  $U \geq T_i$  ( $i=1,2,\dots$ ) and  $(Ux)_n \neq \sup \{(T_i x)_n : i=1,2,\dots\} = x_n$ , then  $U$  may be replaced by the compact linear operator  $U'$ , where

$$\begin{aligned} (U'x)_m &= (Ux)_m & \text{if } m \neq n, \\ &= x_n & \text{if } m = n. \end{aligned}$$

Then  $U \geq U' \geq T_i$ , so that  $U$  is not the supremum of  $\{T_i\}_{i=1}^{\infty}$ .

I.e. if  $\bigvee_{i=1}^{\infty} T_i$  existed it would be the natural injection of  $\ell_1$  into  $\ell_{\infty}$ , which is certainly not compact. Hence  $K(\ell_1, \ell_{\infty})$  cannot be a complete vector lattice.

## Chapter IV

Injectivity and projectivity.

We discuss here some applications of category theory in our context. In the first section we give a brief account of the concepts involved in defining injectivity, projectivity and anti-isomorphism. The second section deals with various categories of interest to us. We determine injective objects in some categories of partially ordered Banach spaces, and projective objects in some categories of compact convex sets.

4.1. Categories.

The account that we give here will follow closely that given by Semadeni in [2].

A category consists of:

- (1) a class  $U^0$ , whose elements are called objects,
- (2) a class  $U$ , whose elements are called morphisms,
- and (3) a law of composition.

These also satisfy the following axioms:

- (4)  $U$  is the union of disjoint sets  $\langle A, B \rangle$ , (one for each ordered pair of objects  $(A, B)$ ).

- (5) The law of composition assigns to each pair  $(\alpha, \beta)$  with  $\alpha \in \langle A, B \rangle$ ,  $\beta \in \langle B, C \rangle$  a morphism  $\gamma \in \langle A, C \rangle$  called the composition of  $\alpha$  and  $\beta$  and denoted by  $\gamma = \beta \alpha$  or  $\beta \circ \alpha$ . This law satisfies the following conditions:

ASSOCIATIVITY : If  $\alpha \in \langle A, B \rangle$ ,  $\beta \in \langle B, C \rangle$  and  $\gamma \in \langle C, D \rangle$ , then  $\gamma(\beta \alpha) = (\gamma \beta)\alpha$ .

EXISTENCE OF IDENTITIES : For each object  $A$ , there is  $1_A \in \langle A, A \rangle$ , called the identity on  $A$ , such that  $\alpha 1_A = \alpha$ ,  $1_A \beta = \beta$  for all  $B \in U^0$ ,  $\alpha \in \langle A, B \rangle$  and  $\beta \in \langle B, A \rangle$ .

If we are dealing with more than one category, and confusion is likely to arise otherwise, we shall write  $\langle A, B \rangle_U$  for a typical set of morphisms. This subscript will be omitted in general. If  $\alpha \in \langle A, B \rangle$  then we call  $A$  the domain of  $\alpha$  and  $B$  its codomain, or range.

Although the definition of a category is mainly in terms of morphisms, we shall name specific categories by both objects and morphisms. E.g. "the category of compact Hausdorff spaces and continuous maps" will be taken to mean the category in which the objects are compact Hausdorff spaces, the morphisms are continuous maps between them, and the law of composition is the usual one. In many cases the specification of the morphisms may be omitted, but we shall not do this, as in the next section we shall on several occasions deal with categories with the same objects but different morphisms.

A category  $U$  is a subcategory of a category  $V$  if the following conditions are satisfied:

- (1)  $U^0 \subset V^0$ ,
- (2)  $U \subset V$ ,
- (3)  $\langle A, B \rangle_U \subset \langle A, B \rangle_V$  for each pair  $(A, B)$  in  $U^0 \times U^0$ ,
- (4) if  $\alpha \in \langle A, B \rangle_U$ ,  $\beta \in \langle B, C \rangle_U$  then their compositions in  $U$  and  $V$  coincide,
- (5) if  $A \in U^0$ , the  $V$ -identity on  $A$  belongs to  $U$  ( and is equal to the  $U$ -identity on  $A$  ).

A subcategory  $U$  of  $V$  is full if  $\langle A, B \rangle_U = \langle A, B \rangle_V$

for any  $A, B \in U^0$ .

A category  $U$  is concrete if there is a transformation  $\square$  mapping  $U^0$  to the class of all sets, satisfying:

(1) for each  $A, B \in U^0$ , the elements of  $\langle A, B \rangle$  are triples  $(\alpha, A, B)$  where  $\alpha$  is a map from  $\square A$  to  $\square B$ ;

(2) the  $U$ -composition of the maps  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  coincides with the ordinary composition of  $\alpha : \square A \rightarrow \square B$  and  $\beta : \square B \rightarrow \square C$ .

All the categories that we shall deal with will be concrete.

If  $U$  is any category, then the dual category  $U^*$  is defined as follows:

(1)  $(U^*)^0 = U^0$ ,

(2) If  $A, B \in U^0$ , then  $\langle A, B \rangle_{U^*} = \langle B, A \rangle_U$ .

(3) If  $\alpha \in \langle A, B \rangle_{U^*}$ ,  $\beta \in \langle B, C \rangle_{U^*}$ , then the  $U^*$ -composition  $\beta * \alpha$  is the  $U$ -composition  $\alpha \beta$ .

$U^*$  is obviously a category, and the identities of  $U^*$  are the same as those of  $U$ . This is purely a formal construction, and a more useful definition will be made later.

A morphism  $\alpha \in \langle A, B \rangle_U$  is a  $U$ -isomorphism ( or shortly an isomorphism ) if and only if there is a morphism  $\beta \in \langle B, A \rangle_U$  such that  $\alpha \beta = \iota_A$  and  $\beta \alpha = \iota_B$ . Such a  $\beta$  is uniquely determined by  $\alpha$ , and is termed the inverse of  $\alpha$ , and is denoted by  $\alpha^{-1}$ . If there is an isomorphism of  $A$  onto  $B$  then  $A$  and  $B$  are termed isomorphic.

A morphism  $\alpha$  in  $U$  is a retraction ( coretraction ) if and only if there exists a morphism  $\beta$  in  $U$  such that  $\alpha \beta$  (  $\beta \alpha$  ) is an identity. If  $\alpha \beta$  is an identity, then  $\beta$  is also called a selection for  $\alpha$ . An object  $B$  is a retract of an object  $A$  if there exists a retraction  $\alpha : A \rightarrow B$ .

A morphism  $\gamma$  is monic or a monomorphism if for any morphisms  $\alpha$  and  $\beta$ ,  $\gamma\alpha = \gamma\beta \implies \alpha = \beta$ ;  $\gamma$  is epic or an epimorphism if for all morphisms  $\alpha$  and  $\beta$ ,

$$\alpha\gamma = \beta\gamma \implies \alpha = \beta.$$

A generator in  $U$  is an object  $G$  such that if  $\alpha, \beta$  are any two morphisms in  $\langle A, B \rangle$ , and  $\alpha \neq \beta$ , then there is a morphism  $\gamma \in \langle G, A \rangle$  such that  $\alpha\gamma \neq \beta\gamma$ . Dually  $G$  is a cogenerator if for each pair of morphisms  $\alpha, \beta \in \langle A, B \rangle$  with  $\alpha \neq \beta$ , there is a morphism  $\gamma \in \langle B, G \rangle$  such that  $\gamma\alpha \neq \gamma\beta$ .

A basic direct ( codirect ) object is a cogenerator ( generator )  $D$  satisfying the following conditions:

(1) if  $G$  is any cogenerator ( generator ), then  $D$  is a retract of  $G$ .

(2) if a cogenerator ( generator )  $G$  is a retract of  $D$ , then  $D$  and  $G$  are isomorphic.

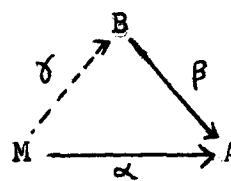
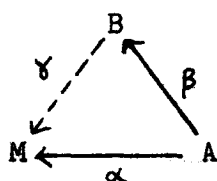
A basic direct ( codirect ) object will be abbreviated to b.d.o. ( b.cd.o. ). If either a b.d.o. or a b.cd.o. exists, then they are unique up to isomorphism.

If  $U$  is a category with a b.d.o.  $D$  ( b.cd.o.  $F$  ), then a submorphism ( supermorphism ) is a morphism  $\alpha \in \langle A, B \rangle$  such that for each morphism  $\beta \in \langle A, D \rangle$  there is a  $\gamma \in \langle B, D \rangle$  such that  $\gamma\alpha = \beta$ , ( for each morphism  $\beta \in \langle F, D \rangle$  there is a  $\gamma \in \langle F, A \rangle$  such that  $\alpha\gamma = \beta$  ). If there is a submorphism  $\alpha \in \langle A, B \rangle$  then  $A$  is termed a subobject of  $B$ .

We are now in a position to define the notions that we really want. An object  $M$  is injective ( projective ) if for every submorphism (supermorphism)  $\xi \in \langle A, B \rangle$  (  $\pi \in \langle B, A \rangle$  ) and for every morphism  $\alpha \in \langle A, M \rangle$  (  $\alpha \in \langle M, A \rangle$  ) there is a morphism  $\beta \in \langle B, M \rangle$  (  $\beta \in \langle M, B \rangle$  ) such that  $\beta\xi = \alpha$

$$(\pi\beta = \alpha).$$

Informally, what we have done is to consider the diagrams:



Firstly we define a simplest non-trivial  $M$ , and then find those  $\beta$  such that for each  $\alpha$  there is a  $\gamma$  with  $\alpha = \gamma\beta$  ( $\alpha = \beta\gamma$ ). Then for such  $\beta$  we find all  $M$  with this property. These  $M$  are the objects of interest.

If  $U$  is a category with a b.d.o.  $D$  ( b.cd.o.  $F$  ), and  $A \in U^0$ , an envelope ( coenvelope ) of  $A$  is a submorphism  $\sigma \in \langle A, B \rangle$  ( a supermorphism  $\pi \in \langle B, A \rangle$  ) such that if  $H \in U^0$  and  $\beta \in \langle B, H \rangle$  ( $\beta \in \langle H, B \rangle$ ) is any  $U$ -morphism such that  $\beta\sigma$  is a submorphism ( $\pi\beta$  is a supermorphism ), then  $\beta$  is a submorphism ( supermorphism ). An injective envelope ( projective coenvelope ) is an envelope ( coenvelope ) such that  $B$  is injective ( projective ). If either of these exists, then it is unique up to isomorphism.

For a typical example of these notions, we may consider the category of compact Hausdorff spaces and continuous maps. The basic codirect object is a one point space, and a supermorphism is any morphism that is onto. Gleason [1], has shown that a space is projective if and only if it is extremally disconnected . He has also shown that every object has a projective coenvelope.

Let  $U$  and  $V$  be categories. A covariant functor from  $U$  to  $V$  consists of two transformations, ( both denoted by the same symbol ) ; the object transformation  $\Phi: U^0 \rightarrow V^0$  assigning to each object  $A$  in  $U^0$  an object  $\Phi(A)$  in  $V^0$  ; and

and a morphism transformation assigning to each  $\alpha \in \langle A, B \rangle_U$  a morphism  $\bar{\Phi}(\alpha) \in \langle \bar{\Phi}(A), \bar{\Phi}(B) \rangle_V$ . These transformations are subject to the two conditions:

$$(1) \text{ if } A \in U^0 \text{ then } \bar{\Phi}(\iota_A) = \iota_{\bar{\Phi}(A)}$$

$$(2) \text{ if } \alpha \in \langle A, B \rangle_U \text{ and } \beta \in \langle B, C \rangle_U \text{ then}$$

$$\bar{\Phi}(\beta\alpha) = \bar{\Phi}(\beta)\bar{\Phi}(\alpha).$$

A contravariant functor from  $U$  to  $V$  consists of a similar object transformation  $\bar{\Phi}: U^0 \rightarrow V^0$ , and a morphism transformation assigning to each  $\alpha \in \langle A, B \rangle_U$  a morphism  $\bar{\Phi}(\alpha) \in \langle \bar{\Phi}(A), \bar{\Phi}(B) \rangle_V$ , that is subject to:

$$(1') \text{ if } A \in U^0 \text{ then } \bar{\Phi}(\iota_A) = \iota_{\bar{\Phi}(A)}$$

$$(2') \text{ if } \alpha \in \langle A, B \rangle_U \text{ and } \beta \in \langle B, C \rangle_U \text{ then}$$

$$\bar{\Phi}(\beta\alpha) = \bar{\Phi}(\alpha)\bar{\Phi}(\beta).$$

Two categories  $U$  and  $V$  are isomorphic ( anti-isomorphic ) if there exists a (1,1) covariant ( contravariant ) functor from  $U$  onto  $V$ . Note that  $U$  is anti-isomorphic to  $V$  if and only if  $U$  is isomorphic to the dual  $V^*$  of  $V$ .

Results that involve only knowledge of the category, as such, can be inferred for a category isomorphic to one for which they are known, as can the dual result for a category anti-isomorphic to one in which the result is known.

#### 4.2. Injective and projective objects.

We look firstly at a very general category. The objects will be all partially ordered Banach spaces with closed, normal and generating cones, and the morphisms will be bounded positive linear operators.

Firstly, we recall the following theorem, 2.2.4 of

Peressini [1].

THEOREM 4.2.1. If  $X, Y, Z$  belong to this category, and  $Z$  is a complete vector lattice, with  $S \in \langle X, Z \rangle$ ,  $T \in \langle X, Y \rangle$ , then there is a morphism  $U \in \langle Y, Z \rangle$  such that  $S = UT$  if and only if the set  $\{Sx : Tx \leq y \text{ for some } y \in Y_1\}$  is bounded above in  $Z$ .

PROPOSITION 4.2.2. This category has a b. d.o., namely the real line with the usual order.

Proof.  $R$  is a cogenerator, since  $\langle X, R \rangle = X_+^*$  generates  $X^*$ , which separates the points of  $X$ , hence so does  $\langle X, R \rangle$ . Any retract of  $R$  is either  $R$  or  $\{0\}$ , and clearly  $\{0\}$  is not a cogenerator.

We must now show that  $R$  is a retract of any  $X$  in the category, and then  $R$  will certainly be a b. d.o. Choose  $x \in X_+$ , with  $\|x\| = 1$ , and define  $\alpha \in \langle R, X \rangle$  by  $\alpha(r) = rx$ . But using Theorem 4.2.1 we can extend the map  $\beta_0$  of  $\alpha(R)$  into  $R$ , that maps  $rx$  into  $r$ , to a morphism  $\beta \in \langle X, R \rangle$ . This is because if  $rx \leq y$  and  $\|y\| \leq 1$ , then either  $rx \leq 0$  or else  $0 \leq rx \leq y$ . As  $X$  is normal the set  $\{r : 0 \leq rx \leq y\}$  is bounded above, and the extension can be done. As  $\beta\alpha$  is clearly the identity on  $R$ ,  $R$  is a retract of  $X$ .

A submorphism of  $X$  into  $Y$  will be any morphism,  $T$ , in  $\langle X, Y \rangle$  such that  $\{Sx : Tx \leq y \text{ for some } y \in Y_1\}$  is bounded above for any  $S \in \langle X, R \rangle$ . Clearly  $T$  must be a homeomorphism and bipositive. I.e. finding out what a subobject of  $Y$  is, may be reduced to finding out for what closed, positively generated subspaces  $X$  of  $Y$ , do all positive linear functionals on  $X$  extend to positive linear functionals on the whole of  $Y$ .

Some characterisations of such subspaces have been given by Riedl [1], and by Fakhoury [2]. However we do not need to know any further characterisation of a subobject in order to determine the injective spaces in this category.

THEOREM 4.2.3. In the category of partially ordered Banach spaces with closed, normal and generating cones, and bounded positive linear operators between them, the injective objects are the finite dimensional lattices.

Proof. It is easily seen that the finite dimensional lattices are injective. Indeed suppose  $X$  is a subobject of  $Y$  and that  $S \in \langle X, Z \rangle$  where  $Z$  may be identified with  $C(F)$ , for  $F$  a finite set. As  $\xi_f \circ S \in \langle X, R \rangle$  for each  $f \in F$ , ( where  $(\xi_f \circ S)(x) = (Sx)(f)$  ) and  $X$  is a subobject of  $Y$ , the set  $\{(\xi_f \circ S)(x) : x \leq y \text{ for some } y \in Y_1\}$  is bounded above, by Theorem 4.2.1. As  $F$  is finite, the set  $\{(Sx) : x \leq y \text{ for some } y \in Y_1\}$  must also be bounded in  $C(F) = Z$ . Thus  $S$  extends to the whole of  $Y$ , and  $Z$  is indeed injective.

We must now show that an injective space is a finite dimensional lattice. Suppose  $Z$  is injective. Let  $Y = \ell_\infty(Z_{1+}^*)$ , and let  $1$  be the constantly one function in  $Y$ . Let  $I$  be the natural injection of  $Z$  into  $Y$ . We claim that  $I$  is a submorphism, for if  $f \in Z_+^*$ ,  $f = \alpha g$  with  $g \in Z_{1+}^*$ ,  $\alpha \geq 0$ . Now the map  $\alpha \times (\text{evaluation at } g)$  extends  $f$  to the whole of  $Y$ .

As  $Z$  is injective, there is  $T \in \langle Y, Z \rangle$  such that  $T \circ I$  is the identity on  $Z$ . Firstly note that  $T1$  is an order unit for  $Z$ . For if  $z \in Z$  then  $Iz \in \beta [-1, 1]$  for some  $\beta \in R_+$ . Then  $z = T(Iz) \in \beta [-T1, T1]$  as claimed. Secondly, suppose

$\{z_\gamma\}$  is a family in  $Z$ , bounded above by  $z_0$ . It follows that  $\{Iz_\gamma\}$  is a family in  $Y$  that is bounded above by  $Iz_0$ , and so has a supremum  $c$ . We claim that  $Tc$  is the supremum in  $Z$  of  $\{z_\gamma\}$ ; for  $Tc \gg T(Iz_\gamma) = z_\gamma$ , and if  $y \gg z_\gamma$  then  $Iy \gg Iz_\gamma$  so that  $Iy \gg c$ . But then  $y = T(Iy) \gg Tc$ , so that  $Tc$  is the supremum, as claimed.

We have established that  $Z$  is equivalent to a space  $C(\Omega)$  for  $\Omega$  Stonian. Now we make use of an example given by Fakhoury [2], of a space  $Y$  in this category, together with a subobject  $X$  of it, such that there is a sequence of elements  $f_n \in X_{1+}^*$ , with the property that any extension  $\tilde{f}_n$  of  $f_n$  to an element of  $Y_+^*$  will have norm at least  $2^n$ . This certainly implies that the set  $\{f_n(x) : x \leq y \text{ for some } y \in Y_1\}$  attains  $2^n$ .

Now suppose  $\Omega$  is infinite. Let  $U_1$  be an open and closed subset of  $\Omega$  which is not empty and has infinite complement, (if this is not possible then  $\Omega$  is finite). Now define a sequence  $(U_n)$  of open and closed sets, by requiring  $U_{n+1}$  to be a subset of  $\Omega \setminus \bigcup_{i=1}^n U_i$  and that  $\Omega \setminus \bigcup_{i=1}^{n+1} U_i$  be infinite. We now define  $S \in \langle X, C(\Omega) \rangle$  as follows:

$$\begin{aligned} (Sx)(w) &= f_n(x)/n & \text{if } w \in U_n, \\ &= 0 & \text{if } w \notin \bigcup_{n=1}^{\infty} U_n. \end{aligned}$$

It is easily verified that  $S \in \langle X, C(\Omega) \rangle$ . However the set  $\{Sx : x \leq y \text{ for some } y \in Y_1\}$  cannot be bounded, so  $C(\Omega)$  is not injective. I.e. the only injective spaces are the finite dimensional lattices, as claimed.

For one of our subsequent results, we need the following sophistication of the Monotone extension theorem, which

was proved by Wright in [1].

**THEOREM 4.2.4. (WRIGHT).** Let  $A(K)$  be an order unit normed space,  $C(\Omega)$  a complete vector lattice, and  $B$  a closed subspace of  $A(K)$  containing the constants. If  $T \in \mathcal{A}(B, C(\Omega))$ , then  $T$  can be extended to  $\tilde{T} \in \mathcal{A}(A(K), C(\Omega))$  in such a way that  $T$  is an extreme point of the set of extensions.

Proof. Let  $W(B_0)$  denote the set of positive extensions of  $T$  to  $B_0$ . We wish to show that the set of extreme points of  $W(A(K))$  is nonempty. Let  $\mathcal{E}$  denote the set of all pairs  $(T_0, B_0)$  such  $B_0$  is a subspace of  $A(K)$  containing  $B$ , and  $T_0$  is an extreme point of  $W(B_0)$ . Define a partial order on  $\mathcal{E}$  by defining  $(T_1, B_1) \leq (T_2, B_2)$  if and only if  $B_1 \subset B_2$  and  $T_2$  extends  $T_1$ .

If  $\mathcal{E}_0$  is a chain in  $\mathcal{E}$ , let  $B_0 = \bigcup \{B' : (T', B') \in \mathcal{E}_0\}$ , and define  $T_0$  on  $B_0$  to agree with each  $T'$  on  $B'$ . As  $\mathcal{E}_0$  is a chain,  $T_0$  is well-defined; we must show that  $T_0$  is an extreme point of  $W(B_0)$ . If  $T_0 = \frac{1}{2}(T_1 + T_2)$ , then  $T_0|_{B'} = T' = \frac{1}{2}(T_1|_{B'} + T_2|_{B'})$ . As  $T'$  is extreme,  $T_1|_{B'} = T_2|_{B'}$ , so clearly  $T_1 = T_2$ , and  $T_0$  is extreme. Now  $(T_0, B_0)$  is an upper bound in  $\mathcal{E}$  for  $\mathcal{E}_0$ . By Zorn's lemma there is a maximal element,  $(\tilde{T}, \tilde{B})$  of  $\mathcal{E}$ .

If  $\tilde{B} \neq A(K)$ , let  $0 \neq a \in A(K) \setminus \tilde{B}$ , and let  $C$  be the subspace of  $A(K)$  spanned by  $\tilde{B}$  and  $a$ . Define  $U : C \rightarrow C(\Omega)$  by:

$$U(b + \lambda a) = \tilde{T}b + \lambda \alpha$$

where  $\alpha = \inf \{\tilde{T}b : \tilde{B} \ni b \geq a\}$ . Clearly  $U \in W(C)$ . To see that  $U$  is extreme, suppose  $U = \frac{1}{2}(U_1 + U_2)$ , with  $U_i \in W(C)$ . As  $T$  is extreme,  $U_1|_{\tilde{B}} = \tilde{T}$ , so that both  $U_1$  and  $U_2$  extend  $T$ . As  $U_1$  and  $U_2$  are positive we must have

$$U_1(a) \leq \inf \{\tilde{T}b : \tilde{B} \ni b \geq a\} = \alpha.$$

I.e.  $\frac{1}{2}(U_1(a) + U_2(a)) \leq U(a)$ . As we actually have equality,

$U_i(a) \subseteq \alpha$ , so that  $U_i = U$ .

This contradicts the maximality of  $(\tilde{T}, \tilde{B})$ , so that in fact  $\tilde{B} = A(K)$ , and the proof is complete.

COROLLARY 4.2.5. With the notation of Theorem 4.2.1, if  $T$  is an extreme point of  $\mathcal{A}(B, C(\mathcal{Q}))$ , then  $T$  can be extended to an extreme point  $\tilde{T}$  of  $\mathcal{A}(A(K), C(\mathcal{Q}))$ .

Proof. We claim that the set of extensions of  $T$  is a face of  $\mathcal{A}(A(K), C(\mathcal{Q}))$ . For if  $\tilde{T}$  extends  $T$ , and  $\tilde{T} = \frac{1}{2}(\tilde{T}_1 + \tilde{T}_2)$ , then clearly both  $\tilde{T}_1|_B$  and  $\tilde{T}_2|_B$  belong to  $\mathcal{A}(B, C(\mathcal{Q}))$ . But  $T$  was extreme, so  $\tilde{T}_1|_B = T$ , and  $\tilde{T}_1$  extend  $T$ . Now an extreme point of this set of extensions will be an extreme point of the set  $\mathcal{A}(A(K), C(\mathcal{Q}))$ .

The first result that we can use this to prove is Gleason's result, [1], that the projective compact Hausdorff spaces are the Stonian spaces. Note firstly that in this category the basic  $\omega$ -direct object is a singleton, so that a supermorphism is simply a map onto. The functor  $C(\quad)$ , assigning to each space  $\mathcal{Q}$ , the Banach space  $C(\mathcal{Q})$ ; and to each continuous map  $\pi : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$  the lattice homomorphism  $C(\pi) : C(\mathcal{Q}_1) \rightarrow C(\mathcal{Q}_2)$  defined by:

$$(C(\pi)f)(w_1) = f(\pi w_1) \quad (f \in C(\mathcal{Q}_2), w_1 \in \mathcal{Q}_1),$$

is a  $(1,1)$  contravariant functor of the category of compact Hausdorff spaces onto the category of order unit normed lattices, with unit-preserving lattice homomorphisms. Hence in this latter category, a submorphism is simply a  $(1,1)$  bipositive morphism. I.e. a subobject is simply a closed subspace containing the order unit.

THEOREM 4.2.6. In the category of order unit normed lattices, with order unit preserving lattice homomorphisms, the injective spaces are precisely the complete lattices.

Proof. That only such spaces are injective follows much as in the third paragraph of the proof of Theorem 4.2.3. To see that these spaces are injective we use Corollary 4.2.5.

COROLLARY 4.2.7. (GLEASON). In the category of compact Hausdorff spaces and continuous maps, the projective spaces are precisely the Stonian spaces.

Proof. The proof uses Theorem 1.1.18 to establish the duality of the two categories, and then Theorem 1.3.5 to relate the complete lattices with the Stonian spaces.

The next result that we wish to present is due to Semadeni [1]. He considered the category of compact convex sets and continuous affine maps between them. Again we make use of the anti-isomorphism of this category with another. In this case  $A(\quad)$ , assigning to each  $K$  the space  $A(K)$ ; and to each  $\pi \in \langle K_1, K_2 \rangle$ , the map  $A(\pi) \in \langle A(K_2), A(K_1) \rangle$ , defined by:

$$(A(\pi)a)(k_1) = a(\pi k_1) \quad (a \in A(K_2), k_1 \in K_1)$$

is a contravariant functor from this category onto the category of order unit normed spaces and positive unit-preserving linear operators. We have immediately, even more easily than the last two results:

THEOREM 4.2.8. In the category of order unit normed spaces and positive unit-preserving linear operators, the injective spaces are precisely the complete vector lattices.

COROLLARY 4.2.9. (SEMADENI). In the category of compact convex sets and continuous affine maps, the projective spaces are precisely the Bauer simplexes whose extreme boundary is a Stonian space.

The next problem that we look at differs from the last two in that we are not actually dealing with a category. We shall in fact look at the collection of all compact convex sets, and extreme continuous affine maps between them. Examples by Davies in Jellet [3], and Lazar [1], show that this is not a category, as the composition of two extreme maps need not be extreme. However the concept of a projective object can still be defined perfectly adequately, if a supermorphism is simply defined to be a map that is onto. Similarly in the collection of order unit normed spaces and extreme positive operators, if a submorphism is defined to be a map that is (1,1) and bipositive, injective objects can be defined.

As before we have:

THEOREM 4.2.10. In the collection of order unit normed spaces and extreme positive operators the injective spaces are precisely the complete vector lattices.

Proof. Corollary 4.2.5 easily shows that these spaces are injective.

The converse needs slightly more care than the previous cases.

Suppose  $A(K)$  is injective. We use the result, (Gleason [1] Theorem 3.2), that every compact Hausdorff space has a projective coenvelope, so that there is a Stonian space  $\Omega$ , and a continuous map  $\pi: \Omega \xrightarrow{\text{onto}} \overline{\partial_e K}$  such that if  $\Omega_0 \subset \Omega$ ,  $\Omega_0$  is closed, and  $\Omega_0 \neq \Omega$ , then  $\pi(\Omega_0) \neq \overline{\partial_e K}$ . It follows that  $\pi^{-1}(\partial_e K)$

is dense in  $\Omega$ , as  $\overline{\pi(\pi^{-1}(\partial_e K))}$  is a closed subset of  $\overline{\partial_e K}$  containing  $\partial_e K$ , and hence is all of it.

It follows that if  $\tilde{\pi} : P(\Omega) \rightarrow K$  is the unique continuous affine extension of  $\pi$ , then  $\tilde{\pi}$  is extreme. For if  $\tilde{\pi} = \frac{1}{2}(\tilde{\pi}_1 + \tilde{\pi}_2)$ , then  $\tilde{\pi}_1 = \tilde{\pi}_2$  on  $\pi^{-1}(\partial_e K)$ , hence on  $\Omega$  by continuity, and thence on  $P(\Omega)$  by continuity and affineness. Now  $A(\tilde{\pi})$  is an extreme positive operator from  $A(K)$  into  $C(\Omega)$  which is  $(1,1)$  and bipositive, so the argument of Theorem 4.2.3 may be repeated.

COROLLARY 4.2.11. In the collection of compact convex sets and extreme continuous affine maps between them, the projective sets are precisely those Bauer simplexes for which the extreme boundary is a Stonian space.

Again, ignoring the fact that these collections do not form categories, we can define the notion of projective coenvelope and injective envelope in the obvious manner. However we have no results in this direction. If we knew that Theorem 1.1.19 held without the metrizability condition, then we could prove the existence of projective coenvelopes in the collection of simplexes and extreme continuous affine maps. As it is however, this result tells us nothing.

Maps that are of rather more interest between compact convex sets, are those that are not only continuous and affine, but also preserve extreme points. As we have stated before, these are extreme, but do not exhaust the ~~extreme~~ maps. It is again easily seen that the b.s.d.o. is a singleton, and that  $\pi$  is a supermorphism if and only if it is onto.

THEOREM 4.2.12. In the category of compact convex sets and extreme point preserving continuous affine maps, the projective elements are precisely the finite dimensional simplexes.

Proof. It is easily verified that the finite dimensional simplexes are in fact projective. The converse is proved in four steps. Firstly we show that if  $K$  is projective, then  $\partial_e K$  is discrete. Secondly, if  $k_0 \in \partial_e K$ , we show that  $\{k_0\}'$ , the union of all the faces of  $K$  that do not contain  $k_0$ , is a closed face. Thirdly we show that  $\partial_e K$  is closed, and finally that  $K$  is a finite dimensional simplex.

Suppose that  $K$  is a projective compact convex subset of  $E$ , and that  $k_0 \in \partial_e K$ . Define  $K_1$  to be the convex hull of the set  $\{(k_0, -1), (k_0, 1)\} \cup K \times \{0\}$  in  $E_1 = E \times \mathbb{R}$  with the product topology. It is easily verified that  $K_1$  is a compact convex set. Also it is easily seen that the extreme boundary of  $K_1$  is the set  $\{(k_0, 1), (k_0, -1)\} \cup (\partial_e K \setminus \{k_0\}) \times \{0\}$ . If  $\pi$  denotes the natural projection of  $K_1$  onto  $K$ , then  $\pi$  is extreme point preserving, continuous and affine. As  $K$  is projective, there is an admissible map  $\varphi$  of  $K$  into  $K_1$  such that  $\pi \varphi = \text{id}_K$ . Clearly if  $k \in \partial_e K \setminus \{k_0\}$  then  $\varphi(k) = (k, 0)$ , as  $(k, 0)$  is the only point of  $K_1$  mapped onto  $k$  by  $\pi$ .

If  $k_0$  is an extreme point of  $K$ , and is not isolated in  $\partial_e K$ , then there is a net  $(k_\gamma)$  in  $\partial_e K \setminus \{k_0\}$  converging to  $k_0$ . As  $\varphi$  is continuous,

$$\varphi(k_0) = \lim \varphi(k_\gamma) = \lim (k_\gamma, 0) = (k_0, 0),$$

which is not an extreme point of  $K_1$ . Hence  $\varphi$  is not a morphism in this category, so there can be no non-isolated points of  $\partial_e K$ .

Let  $F = \{k_0\}'$ . Then  $F \subset \overline{\text{co}}(\partial_e K \setminus \{k_0\})$ , for by the

integral form of the Krein-Milman Theorem, if  $p \in F$  there is a Borel probability measure  $\mu$  representing  $p$  and supported by  $\overline{\partial_e K}$ . If  $\mu(\{k_0\})$  were non-zero, then  $p$  could be written as a proper convex combination of  $k_0$  and another point, which is impossible as  $p$  belongs to a face of  $K$  not containing  $k_0$ . It follows that  $\mu$  is supported by  $\overline{\partial_e K \setminus \{k_0\}} = \overline{\partial_e K} \setminus \{k_0\}$ , so that  $p \in \overline{\text{co}(\partial_e K \setminus \{k_0\})}$ . Also this latter set does not contain  $k_0$ , as this point is isolated amongst the extreme points.

As  $\varphi$  is continuous and affine,  $\varphi(f) = (f, 0)$  for each  $f \in F$ .

We also have  $\varphi(k_0) = (k_0, 1)$  (say). Suppose that  $f_1, f_2 \in F$

and  $(f_1 + f_2)/2 \notin F$ . For some  $\lambda > 1$ , the point

$k_0 + \lambda((f_1 + f_2)/2 - k_0)$  belongs to  $F$ , since  $K = \text{co}(F \cup \{k_0\})$ .

It follows that

$$\varphi(k_0 + \lambda((f_1 + f_2)/2 - k_0)) = (k_0 + \lambda((f_1 + f_2)/2 - k_0), 0).$$

But we also have:

$$\begin{aligned} \varphi(k_0 + \lambda((f_1 + f_2)/2 - k_0)) &= (1-\lambda)\varphi(k_0) + (\lambda/2)\varphi(f_1) + (\lambda/2)\varphi(f_2) \\ &= ((1-\lambda)k_0 + (\lambda/2)(f_1 + f_2), 1-\lambda) \end{aligned}$$

Thus  $\lambda = 1$ , and  $F$  is convex and hence a face.

Now suppose that  $F$  is not closed. If  $p \in \overline{F}$ , then

$p = \alpha q + (1-\alpha)k_0$  with  $0 \leq \alpha < 1$  and  $q \in F$ , so that we have

$$\begin{aligned} \varphi(p) &= \varphi(\alpha q + (1-\alpha)k_0) \\ &= \alpha\varphi(q) + (1-\alpha)\varphi(k_0) \\ &= \alpha(q, 0) + (1-\alpha)(k_0, 1) \\ &= (\alpha q + (1-\alpha)k_0, \alpha). \end{aligned}$$

But  $p \in \overline{F}$ , and so  $\varphi(p) = (p, 0)$  as  $\varphi$  is continuous. Thus  $\alpha = 1$ , and  $F = \overline{F}$ .

Let  $k_1 \in \overline{\partial_e K}$ , then for each  $k_0 \in \partial_e K$ , either  $k_1 = k_0$  or

$k_1 \in \{k_0\}'$ , by what we have just proved. Consider the intersection of the sets  $\{k_0\}'$  as  $k_0$  ranges over  $\partial_e K$ . This is a closed face of  $K$ . If it were non-empty it would possess extreme points, which would also be extreme points of  $K$  since this is a face of  $K$ . However it contains none of these by construction, and is therefore empty. It follows that  $\partial_e K$  is closed.

Finally since  $\partial_e K$  is discrete and compact, it is a finite set. Let  $K$  have  $n+1$  extreme points, and let  $S_n$  be an  $n$ -dimensional simplex. Let  $\pi : S_n \rightarrow K$  be any affine map which maps the set of extreme points of  $S_n$  onto those of  $K$ .  $\pi$  is admissible and onto, and so there is an admissible map  $\varphi : K \rightarrow S_n$  such that  $\pi \varphi = \text{id}_K$ . We now have

$$\begin{aligned} \dim(\text{aff}(K)) &\geq \dim(\text{aff}(\varphi(K))) \\ &= \dim(\text{aff}(S_n)) \quad (\text{as } \varphi \text{ must be onto}) \\ &= n. \end{aligned}$$

Hence  $K$  is an  $n$ -dimensional simplex, and the proof is complete.

COROLLARY 4.2.13. In the category of simplexes and extreme point preserving continuous affine maps, the projective objects are the finite dimensional simplexes.

Proof. To prove this we need only verify that the set  $K_1$  constructed in the proof of Theorem 4.2.12 is a simplex whenever  $K$  is. To see this we use the fact that a compact convex set  $K$  is a simplex if and only if  $A(K)$  has the R.D.P., (Proposition 1.3.2). The space of continuous affine functions on  $K_1$  is order isomorphic to the space,  $B$ , of all triples  $(a, x', x'')$  with  $a \in A(K)$ ,  $x', x'' \in \mathbb{R}$  such that  $2a(k_0) = x' + x''$ , when we define  $(a, x', x'') \geq (0, 0, 0)$  if and only if  $a \geq 0$  and  $x', x'' \geq 0$ .

Suppose that we have

$$(a, x', x''), (b, y', y'') \geq (0, 0, 0) \quad (1)$$

and

$$(a, x', x'') + (b, y', y'') \geq (c, z', z'') \geq (0, 0, 0). \quad (2)$$

Define  $g(k_0) = \inf \{ a(k_0), c(k_0), \frac{1}{2}(x' + x''), \frac{1}{2}(x'' + x') \}$   
 $g(k) = \inf \{ a(k), c(k) \} \quad (k \in K \setminus \{k_0\})$

and

$$h(k) = \sup \{ c(k) - b(k), 0 \} \quad (k \in K \setminus \{k_0\})$$

$$h(k_0) = \sup \{ c(k_0) - b(k_0), 0, \frac{1}{2}(z'' - y''), \frac{1}{2}(z' - y') \}.$$

By Theorem 1.3.6 there is  $c_1 \in A(K)$  with  $h \leq c_1 \leq g$ . Putting

$c_2 = c - c_1$  we see that  $a \geq c_1 \geq 0$ ,  $b \geq c_2 \geq 0$  and  $c_1 + c_2 = c$ ;

and also that  $2c_1(k_0) \geq z'' - y''$ ,  $z' - y'$ . This, together with

(1) and (2) tells us that:

$$x', 2c_1(k_0), z', 2c_1(k_0) + y'' - z'' \geq 0, 2c_1(k_0) - x'', z' - y', 2c_1(k_0) - z''.$$

Let  $z_1'$  separate the two sides of this inequality, and define

$$z_2' = z' - z_1', z_1'' = 2c_1(k_0) - z_1', \text{ and } z_2'' = 2c_2(k_0) - z_2'. \text{ It is}$$

then clear that  $(c_1, z_1', z_1'') \in B$ , and that we have

$$(c_1, z_1', z_1'') + (c_2, z_2', z_2'') = (c, z', z'').$$

From the definition of  $z_1'$  and the definitions it is easily

verified that

$$(a, x', x'') \geq (c_1, z_1', z_1'') \geq (0, 0, 0),$$

$$(b, y', y'') \geq (c_2, z_2', z_2'') \geq (0, 0, 0).$$

Thus  $B$  has the R.D.P. and  $K_1$  is a simplex, concluding the proof.

COROLLARY 4.2.14. In the category of order unit normed spaces with the R.D.P., and R-homomorphisms, the injective spaces are the finite dimensional ones.

Proof. Theorem 1.1.20 states precisely that this category is anti-isomorphic to that category considered in the last Corollary,

from which the result is immediate.

We now turn to some categories of caps and continuous affine maps between them which preserve the origin. Again we deal with the three cases of all such maps, the extreme ones, and the extreme point preserving maps. The first result we note is the following.

LEMMA 4.2.15. In all three cases, if  $C$  is projective, and  $C$  is embedded in  $A_0(C)^*$  in the usual manner, then the face  $\{k \in C : \|k\| = 1\}$  complementary to the origin, is closed, ( for the given topology = weak\*-topology ).

Proof. Let  $C_1$  be the cap  $\text{co}(C \times \{1\} \cup \{(0,0)\})$  in  $A_0(C)^* \times \mathbb{R}$ . Let  $\pi$  be the natural projection of  $C_1$  onto  $C$ , then  $\pi$  is a morphism in all the cases that we consider. As  $C$  is assumed to be projective, there is a morphism  $\varphi : C \rightarrow C_1$ , with  $\pi \varphi = \iota_C$ . We clearly have  $\varphi(k) = (k,1)$  if  $\|k\| = 1$ . The argument used in Theorem 4.2.12 now gives us the result.

By limiting ourselves to caps which do have this property, we see that this face must be projective for the appropriate collection of compact convex sets. On the other hand, suppose this face  $F$  of  $C$  is closed, and projective as a compact convex set. Let  $\pi : C_2 \rightarrow C_1$ ,  $\psi : C \rightarrow C_1$  be any other two morphisms with  $\pi$  onto. We can find  $\varphi : F \rightarrow C_2$  such that  $\pi \varphi = \psi|_F$ , which is a morphism in the collection. We can now extend  $\varphi$  uniquely to be continuous and affine on  $C$  and preserve the origin. We still have  $\pi \varphi = \psi$ , as required, and it is easily verified that  $\varphi$  still has the required properties.

We can thus state without further proof the following three results:

COROLLARY 4.2.16. In the category of caps and continuous affine maps preserving the origin, the projective caps are those for which the face complementary to the origin is a ( compact ) Bauer simplex whose extreme boundary is a Stonian space.

COROLLARY 4.2.17. In the collection of caps and extreme continuous affine maps preserving the origin, the projective caps are those for which the face complementary to the origin is a ( compact ) Bauer simplex whose extreme boundary is a Stonian space.

COROLLARY 4.2.18. In the category of caps and extreme point preserving continuous affine maps that preserve the origin, the projective caps are the finite dimensional simplexes.

The anti-isomorphic categories obtained in the usual way have as objects a.o.u.-normed spaces. The morphisms are positive linear operators of norm at most one. We can state automatically from the previous results:

COROLLARY 4.2.19. In the category of a.o.u.-normed spaces, and all positive linear operators of norm at most one, the injective spaces are the order unit normed complete vector lattices.

COROLLARY 4.2.20. In the collection of a.o.u.-normed spaces, and extreme operators in the set of those that are positive and of norm at most one, the injective spaces are the order unit normed complete vector lattices.

## Chapter V

Facial topologies and related topics.

In this chapter we look briefly at facial topologies on compact convex sets, and other ways in which the various notions involved arise. Section one deals with the definitions and elementary properties of facial topologies. In the second section we generalise a theorem of Lloyd on subalgebras of  $A(K)$ , and show how parallel faces arise in a generalisation of the Banach-Stone theorem. In the third section we show how split faces arise in the decomposition of a space  $A(K)$  into a certain kind of direct sum.

5.1 Facial topologies.

Historically, facial topologies were first employed by Effros in his study of simplex spaces ( see Effros [ 1 ] and [ 2 ], and Effros and Gleit [ 1 ] ). However we will present the theory only for spaces  $A(K)$ .

If  $K$  is a compact convex set, and  $F$  is a face of  $K$ , then  $F'$ , the complementary  $\sigma$ -face of  $K$ , is the union of all the faces of  $K$  not meeting  $F$ . If we further have that  $F$  is closed, then  $K = \text{co}(F \cup F')$ . A proper face  $F$  of  $K$  is termed a split face of  $K$  if  $F'$  is a face, and each point of  $K \setminus (F \cup F')$  can be uniquely expressed as a convex combination of a point of  $F$  and one of  $F'$ . The empty set,  $\phi$ , and  $K$  are termed improper split faces.

The split faces of compact convex sets are of interest because of the following result:

PROPOSITION 5.1.1. (ALFSEN and ANDERSEN). The collection of closed split faces of a compact convex set  $K$  is closed under finite convex hulls and arbitrary intersections.

As a result of this we see that the collection of all sets  $F \cap \partial_e K$ , for  $F$  a closed split face of  $K$ , form the closed sets for a topology on  $\partial_e K$ , the  $\mathbb{Q}$ -topology. It is easily verified that this topology is compact, but not necessarily Hausdorff. Much of the usefulness of this topology stems from the following:

THEOREM 5.1.2. (ALFSEN and ANDERSEN). If  $f : \partial_e K \rightarrow \mathbb{R}$  is continuous for the  $\mathbb{Q}$ -topology, then there is a unique  $a \in A(K)$  which extends  $f$ .

The centre of a space  $A(K)$  is the vector subspace of all  $a \in A(K)$  with the property that, for each  $b \in A(K)$  there is  $c \in A(K)$  such that

$$a(k)b(k) = c(k) \quad (\forall k \in \partial_e K).$$

We denote this space by  $\tilde{A}(K)$ .

THEOREM 5.1.3. (ALFSEN and ANDERSEN). Let  $a \in A(K)$ , then  $a \in \tilde{A}(K)$  if and only if  $a|_{\partial_e K}$  is continuous for the  $\mathbb{Q}$ -topology.

If  $K$  is a simplex, then every closed face is split, (Alfsen [1]). In this case there are a number of other characterisations of  $\tilde{A}(K)$ , which are not valid in general. For instance  $\tilde{A}(K)$  is the largest subalgebra of  $A(K)$ , for the operation of point-wise multiplication on  $\partial_e K$ . Also  $\tilde{A}(K)$  is the largest vector sublattice of  $A(K)$  containing the constants. In order to extend this last result, the concept of facial topologies has been further extended by Rogalski.

A closed face  $F$  of  $K$  is termed a parallel face of  $K$  if:

- (1)  $F'$  is a face;
- (2) each element of  $K \setminus (F \cup F')$  can be written as  $\lambda f + (1 - \lambda)f'$  with  $f \in F$ ,  $f' \in F'$  and  $0 < \lambda < 1$ ;
- and (3) if an element of  $K \setminus (F \cup F')$  has two such representations  $\lambda f + (1 - \lambda)f'$  and  $\lambda_1 f_1 + (1 - \lambda_1)f'_1$ , then  $\lambda = \lambda_1$ .

PROPOSITION 5.1.4. (ROGALSKI). If  $(F_\alpha)$  is a downward filtering family of parallel faces of  $K$ , then  $F = \bigcap_\alpha F_\alpha$  is a parallel face of  $K$ .

A family  $\mathcal{F}$  of parallel faces of  $K$  is termed topological if it is closed for the operations of finite intersections, finite convex hulls, and downward filtering intersections and contains  $\phi$  and  $K$ . We then have:

PROPOSITION 5.1.5. (ROGALSKI). If  $\mathcal{F}$  is a topological family of parallel faces of  $K$ , the sets  $F \cap \partial_e K$  for  $F \in \mathcal{F}$  are the closed sets of a compact topology on  $\partial_e K$ .

Such a topology is termed a facial topology. These topologies share the extension property of Theorem 5.1.2.

THEOREM 5.1.6. (GOULLET de RUGY). Any facially continuous function  $f$  extends to a unique  $a \in A(K)$ .

The set of all such extensions, taken over all possible facial topologies, is termed the weak centre of  $A(K)$ . We will denote it by  $\hat{A}(K)$ . It is of interest to note firstly that there are facial topologies, for instance the  $\mathbb{Q}$ -topology. Secondly there are other such topologies. Consider, for example, a square  $\square$  in  $\mathbb{R}^2$ . Its split faces are only  $\phi$  and  $\square$ , so that

$\tilde{A}(\square)$  comprises solely the constant functions. On the other hand any edge of the square is a parallel face, and a pair of opposite edges together with  $\phi$  and  $\square$  constitute a topological family. Thus  $\hat{A}(\square)$  consists of all continuous affine functions on  $\square$  which are constant on some edge. Note that this example shows that in general  $\hat{A}(K)$  need not be a vector space.

One characterisation of  $\hat{A}(K)$  is the following. Another will be given in the next section.

PROPOSITION 5.1.7. (ROGALSKI). Let  $K$  be a compact convex set,  $a \in A(K)$ . The following are equivalent:

- (a)  $a \in \hat{A}(K)$ .
- (b) For any finite family of pairs of reals  $(\lambda_p, \mu_p)$ , the supremum  $\bigvee_{p=1}^n (\lambda_p a + \mu_p 1_K)$  exists in  $A(K)$ .

Accounts of the results in this section may be found in Alfsen and Andersen [1] and [2], and Rogalski [1]. As we stated previously, the study of facial topologies was initiated by Effros in a rather more general context. In [3] he extended the theory even further, to those (unordered) real Banach spaces whose dual is an  $L^1(\mu)$ -space. These spaces have been classified by Lindenstrauss and Wulbert [1], and some properties studied by Lacey and Morris [1]. Fakhoury in [1] has continued the study of facial topologies in this context and has a notion of centre for these spaces.

## 5.2. Subalgebras of $A(K)$ and the Banach-Stone theorem.

Lloyd has shown, in [1], that if  $K$  is a simplex, and  $B$  is a subspace of  $A(K)$  that is an algebra under the operation

of pointwise multiplication on  $\partial_e K$ , then  $B \subset \tilde{A}(K)$ . He also gives an example to show that this is not true without the assumption that  $K$  is a simplex. (The square in  $\mathbb{R}^2$  mentioned earlier provides a simple example.)

We can however prove a similar result, replacing  $\tilde{A}(K)$  by the space  $\hat{A}(K)$ , and in doing so obtain another characterisation of this space.

THEOREM 5.2.1. Let  $K$  be a compact convex set, and  $B$  a subspace of  $A(K)$  which is an algebra for pointwise multiplication on  $\partial_e K$ . Then  $B \subset \hat{A}(K)$ .

Proof. Firstly we note that we may assume that  $B$  contains the constants, (since  $B + \mathbb{R}1_K \subset A(K)$  and if  $a, b \in B$ ;  $\lambda, \mu \in \mathbb{R}$ , then  $(a + \lambda 1_K)(b + \mu 1_K) = (ab + \mu a + \lambda b) + \lambda \mu 1_K \in B + \mathbb{R}1_K$ ), and is closed (since  $A(K)$  is complete). Note also that the algebra operations are pointwise operations on  $\overline{\partial_e K}$ , by continuity of all the functions involved.

Define an equivalence relation  $Q$  on  $\overline{\partial_e K}$  by  $xQy$  if and only if  $a(x) = a(y)$  for all  $a \in B$ . Let  $\overline{\partial_e K}/Q$  be given the quotient topology, so that an application of the Stone-Weierstrass theorem, and noting that  $a \in B$  attains its maximum modulus on  $\partial_e K$ , shows that  $B$  is norm and order-isomorphic to  $C(\overline{\partial_e K}/Q)$ . Let  $\pi$  denote the natural projection of  $\overline{\partial_e K}$  onto  $\overline{\partial_e K}/Q$ . The sets  $\pi^{-1}(D)$ , for  $D$  closed in  $\overline{\partial_e K}/Q$ , form the closed sets for a topology  $\mathcal{T}$  on  $\overline{\partial_e K}$ , which is easily verified to be compact although not necessarily Hausdorff. For this topology  $B$  may be identified with  $C(\overline{\partial_e K})$ . Consider the relativisation of this topology to  $\partial_e K$ . We shall show that this is a facial topology, and as each function in  $B$  will certainly be continuous

for this topology, we must have  $B \subset \hat{A}(K)$ .

Suppose  $D$  is relatively  $\mathcal{J}$ -closed in  $\partial_e K$ , so that  $D = D_0 \cap \partial_e K$  for  $D_0$   $\mathcal{J}$ -closed in  $\overline{\partial_e K}$ . In particular  $D_0$  is closed for the given topology on  $K$ . We firstly show that  $\overline{\text{co}}(D)$  is a face of  $K$ .

Suppose  $d \in \overline{\text{co}}(D)$ , and that  $d$  is represented by a Borel probability measure  $\mu$  supported by  $\overline{\partial_e K}$ , (by the integral form of the Krein-Milman theorem.). Let  $k \in \overline{\partial_e K} \setminus D_0$ . As  $\overline{\partial_e K}/Q$  is Hausdorff and  $D_0/Q$  is closed, there is a function  $f \in C(\overline{\partial_e K}/Q)$  such that:

$$(1) f|_{D_0/Q} \equiv 0,$$

$$(2) f(\pi(k)) = 1,$$

$$\text{and } (3) 1 \geq f \geq 0.$$

Then there is an open neighbourhood of  $\pi(k)$ ,  $U$ , such that  $f|_U > \frac{1}{2}$ . As we can identify  $B$  with  $C(\overline{\partial_e K}/Q)$ , there is an  $a \in B$  such that:

$$(1) a|_{D_0} \equiv 0,$$

$$(2) a|_{\pi^{-1}(U)} > \frac{1}{2},$$

$$\text{and } (3) 1 \geq a \geq 0.$$

As  $a(d) = 0$  (by continuity and affineness of  $a$ ),  $\mu$  can have no mass on the open set  $\pi^{-1}(U)$ , containing  $k$ . As  $k$  was an arbitrary point of  $\overline{\partial_e K} \setminus D_0$ ,  $\text{supp}(\mu) \subset D_0$ .

Now if  $d = \frac{1}{2}(d_1 + d_2)$ , each  $d_i$  can be represented by a Borel probability measure  $\mu_i$  supported by  $\overline{\partial_e K}$ . As  $\frac{1}{2}(\mu_1 + \mu_2)$  represents  $d$ , it is supported by  $D_0$ . Since both  $\mu_1$  and  $\mu_2$  are positive, both  $\mu_1$  and  $\mu_2$  are in fact supported by  $D_0$ , so that  $d_i \in \overline{\text{co}}(D)$ . I.e. we have shown that  $\overline{\text{co}}(D)$  is a face of  $K$ .

That the relative  $\mathcal{J}$ -topology is compact is now easily

seen. Let  $(D_\alpha)$  be a family of  $\mathcal{J}$ -relatively closed subsets of  $\partial_e K$ , with the finite intersection property. The family of closed faces  $\overline{\text{co}}(D_\alpha)$  also has the finite intersection property, so has a non-empty intersection,  $K$  being compact. As  $E = \bigcap_\alpha \overline{\text{co}}(D_\alpha)$  is a closed face of  $K$ , and we have just shown that it is non-empty,  $\partial_e E$  is non-empty. Since  $E$  is a closed face of each  $\overline{\text{co}}(D_\alpha)$ ,  $\partial_e E \subset \partial_e(\overline{\text{co}}(D_\alpha)) = D_\alpha$  for all  $\alpha$ . Hence  $\bigcap_\alpha D_\alpha \supset \partial_e E \neq \emptyset$ .

In particular we now know that  $\partial_e K/Q$  is a compact Hausdorff space. Let  $I$  be the injection of  $C(\partial_e K/Q)$  into  $B \subset A(K)$ . Let  $\tilde{\pi}$  be the associated continuous affine map of  $K$  into  $P(\partial_e K/Q)$ . It is clear that  $\tilde{\pi}|_{\partial_e K} = \pi|_{\partial_e K}$ , and that  $\overline{\text{co}}(\pi^{-1}(D)) = \tilde{\pi}^{-1}(\overline{\text{co}}(D))$  for  $D$  closed in  $\partial_e K/Q$ . Also  $\tilde{\pi}$  is extreme point preserving. Let  $D$  be closed in  $\partial_e K/Q$ , and  $F = \overline{\text{co}}(D)$ , with  $F'$  its complementary face. It is then easily seen that  $\tilde{\pi}^{-1}(F')$  is a face of  $K$  complementary to  $\tilde{\pi}^{-1}(F)$ , and in fact that  $\tilde{\pi}^{-1}(F)$  is a parallel face. Hence the topology on  $\partial_e K$  is easily seen to be a facial topology, and the proof is complete.

COROLLARY 5.2.2. If  $K$  is a compact convex set, then  $\hat{A}(K)$  is precisely the union of all subspaces of  $A(K)$  which are algebras for point-wise multiplication on  $\partial_e K$ .

We now turn to another case in which parallel faces arise in a natural manner.

The classical Banach-Stone theorem states that if  $\Omega_1$  and  $\Omega_2$  are compact Hausdorff spaces, and  $C(\Omega_1)$  is linearly isometric to  $C(\Omega_2)$ , then  $\Omega_1$  is homeomorphic to  $\Omega_2$ . This has been extended to spaces of continuous affine functions on simplexes by Jellet [1] and Lazar [3]. In this case,

if  $A(K_1)$  is linearly isometric to  $A(K_2)$ , then  $K_1$  and  $K_2$  are affinely homeomorphic. This remains true if either  $K_1$  is known to be a simplex, and the other is a general compact convex set. However this is not true in general. To see this we first need a lemma.

LEMMA 5.2.3. Let  $K$  be a compact convex set,  $C$  a closed subset of  $K$  such that  $K \subset \text{co}(C)$ , and  $E$  a locally convex Hausdorff topological vector space. If  $a$  is a <sup>continuous</sup> function from  $C$  into  $E$  such that there is a unique affine extension  $\tilde{a}$  of  $a$  to the whole of  $K$ , then  $\tilde{a}$  is continuous.

Proof. Let  $\Phi : C \times C \times [0,1] \rightarrow K$  map  $(y,z,\lambda) \mapsto \lambda y + (1-\lambda)z$ , and  $\Psi : C \times C \times [0,1] \rightarrow E$  map  $(y,z,\lambda) \mapsto \lambda a(y) + (1-\lambda)a(z)$ . Clearly both  $\Phi$  and  $\Psi$  are continuous. The assumption that  $a$  has a unique affine extension to  $K$ ,  $\tilde{a}$ , amounts to saying that the set  $\Psi(\Phi^{-1}(y,z,\lambda))$  is a singleton for each point of  $C \times C \times [0,1]$ .

If  $D$  is a closed subset of  $E$ , then  $\tilde{a}^{-1}(D) = \Phi(\Psi^{-1}(D))$  is closed. This is because  $\Psi$  is continuous so that  $\Psi^{-1}(D)$  is closed in  $C \times C \times [0,1]$  and is hence compact. Then  $\Phi(\Psi^{-1}(D))$  is compact in  $K$ , and hence closed. Thus  $\tilde{a}$  is continuous as claimed.

PROPOSITION 5.2.4. Let  $K$  be a compact convex set, and  $F_1$  and  $F_2$  disjoint, closed, complementary parallel faces of  $K$ . Let  $k_0$  be a point of the affine span of  $F_2$ , and let  $F_2'$  be the compact convex set  $\{2k_0 - f_2 : f_2 \in F_2\}$ . If  $K' = \text{co}(F_1 \cup F_2')$ , then  $K'$  is a compact convex set and  $A(K')$  is linearly isometric to  $A(K)$ .

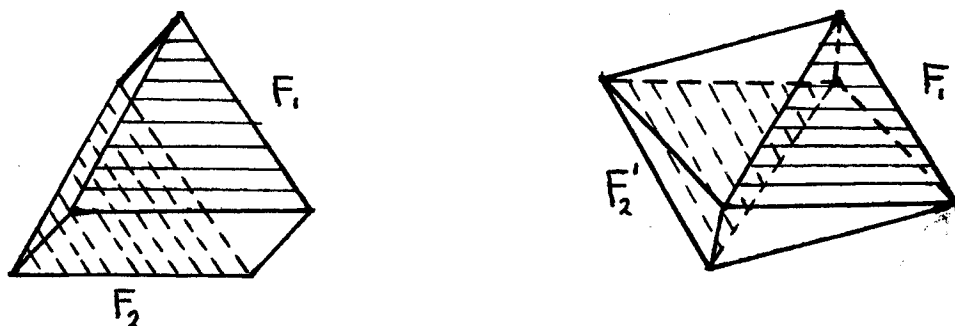
Proof. It is clear that  $K'$  is a compact convex set. Define

$T : A(K) \longrightarrow A(K')$  by:

$$\begin{aligned} Ta|_{F_1} &= a|_{F_1} \\ Ta(2k_0 - f_2) &= -a(f_2), \end{aligned}$$

and  $Ta$  is affine. This exists as  $F_2$  and  $F_1$  are parallel, so that the function defined on  $F_2 \cup F_1$  extends to an affine function on  $K'$ , which is continuous by Lemma 5.2.3. Clearly  $T$  is onto, and an isometry since each  $a$  attains its maximum modulus on  $F_2 \cup F_1$ .

If the faces  $F_1$  and  $F_2$  are split, then  $K$  and  $K'$  are affinely homeomorphic. That this is not always so is seen by considering the convex hull in  $\mathbb{R}^3$  of a triangle with a parallel congruent triangle. This set has 5 faces, whereas the corresponding set  $K'$  will have 8 faces.



What is of interest to us is that this is the only way in which such isometries arise.

THEOREM 5.2.5. Let  $K, K'$  be compact convex sets such that  $A(K)$  is linearly isometric to  $A(K')$ . Then there exist disjoint, complementary, closed parallel faces of  $K$ ,  $F_1$  and  $F_2$ , such that for any point  $k_0 \in F_2$ ,  $K'$  is affinely homeomorphic to the set  $\text{co}(F_1 \cup (2k_0 - F_2))$ .

Proof. We know that  $K$  and  $K'$  are affinely homeomorphic to weak  $*$ -closed faces of the unit ball of  $A^*$  ( $= A(K)^* = A(K')^*$ ), such that  $\text{co}(K \cup -K) = \text{co}(K' \cup -K') = A_1^*$ . It will suffice to prove that any two such faces of  $A_1^*$  are related in the manner described.

Firstly we claim that  $K = \text{co}((K \cap K') \cup (K \cap -K'))$ . If  $k \in \partial_e K$ , then  $k \in \text{co}(K' \cup -K')$  by assumption. But  $K$  is a face of  $A_1^*$ , so  $k$  is an extreme point of  $A_1^*$ , hence  $k \in K' \cup -K'$ . Thus  $\partial_e K \subset (K \cap K') \cup (K \cap -K')$ . Now the Krein-Milman theorem assures us that  $K$  is contained in the compact convex set  $\text{co}((K \cap K') \cup (K \cap -K'))$ .

Let  $F_1 = K \cap K'$  and  $F_2 = K \cap -K'$ . It is clear that  $K' = \text{co}(F_1 \cup -F_2)$ . The claimed result will be proven when we show that  $F_1$  and  $F_2$  are closed parallel faces, for then there will be an affine map of  $\text{co}(F_1 \cup -F_2)$  onto  $\text{co}(F_1 \cup (2k_0 - F_2))$ , which will be continuous by Lemma 5.2.3, and is clearly (1,1).

It is clear that each  $F_i$  is a closed face. We need only show that they are parallel. Consider the function that is identically 1 on  $K'$ . On  $F_1$  this is identically 1, and on  $F_2$  is identically -1. If we have an identity:

$$\lambda f_1 + (1-\lambda)f_2 = \lambda' f'_1 + (1-\lambda')f'_2$$

with  $f_1, f'_1 \in F_1$ ;  $f_2, f'_2 \in F_2$ ; and  $0 < \lambda, \lambda' < 1$ , then composing with this function we see that:

$$\lambda - (1-\lambda) = \lambda' - (1-\lambda').$$

I.e.  $\lambda = \lambda'$ , so that the faces are parallel.

COROLLARY 5.2.6. If either  $K$  or  $K'$  is a simplex, and  $A(K)$  is linearly isometric to  $A(K')$ , then  $K$  and  $K'$  are affinely homeomorphic.

Proof. In this case  $F_1$  and  $F_2$  are automatically split, so there is a (1,1) affine map of  $K = \text{co}(F_1 \cup F_2)$  onto  $\text{co}(F_1 \cup -F_2) = K'$ , which is continuous by Lemma 5.2.3.

Clearly this result still holds if it is known that all the closed faces of either  $K$  or  $K'$  are split. This provides a mild, although not very useful, generalisation of this Corollary.

Similar results are available if we deal with spaces  $A_0(C)$ , of continuous affine functions, vanishing at the origin, on a universal cap. Analogous to Proposition 5.2.4 we have:

PROPOSITION 5.2.7. Let  $C$  be a universal cap, and  $F_1, F_2$  closed parallel faces of  $C$  such that  $C = \text{co}(F_1 \cup F_2)$ ,  $F_1 \cap F_2 = \{0\}$ . If  $C' = \text{co}(F_1 \cup -F_2)$  then  $A_0(C)$  is linearly isometric to  $A_0(C')$ .

Analogous to Theorem 5.2.5 we have:

THEOREM 5.2.8. Let  $C$  and  $C'$  be universal caps such that  $A_0(C)$  and  $A_0(C')$  are linearly isometric. Then  $C$  and  $C'$  can be embedded in an l.c.s. in such a way that there are closed parallel faces  $F_1, F_2$  of  $C$  with  $C = \text{co}(F_1 \cup F_2)$ ,  $F_1 \cap F_2 = \{0\}$ , and such that  $C' = \text{co}(F_1 \cup -F_2)$ .

Proof. The l.c.s. is, of course,  $A$  with the weak\*-topology.

The only difference in the proof is in proving that the faces  $F_1$  and  $F_2$  are parallel faces. The function that is 1 on the face  $\{k \in C' : \|k\| = 1\}$  of  $C'$  and 0 at the origin is now a bounded (but not necessarily continuous) affine function on  $C'$ . The associated function on  $C$  is 1 on  $F_1 \cap \{k \in C : \|k\| = 1\}$  and is -1 on  $F_2 \cap \{k \in C : \|k\| = 1\}$ . Note also that the norm is a bounded affine function on  $C$  that is 0 at the origin

and is 1 at  $\{k \in C : \|k\| = 1\}$ .

The face  $F_2 = \{k \in C : \|k\| = 1\}$  is clearly the face complementary to  $F_1$ . Suppose that we have:

$$\lambda f_1 + (1-\lambda)f_2 = \lambda' f'_1 + (1-\lambda')f'_2$$

with  $f_1, f'_1 \in F_1$ ;  $f_2, f'_2 \in F_2 \cap \{k \in C : \|k\| = 1\}$ ; and

$0 < \lambda, \lambda' < 1$ . Applying our two affine functions, we have:

$$\lambda \|f_1\| - (1-\lambda) = \lambda' \|f'_1\| - (1-\lambda'),$$

$$\lambda \|f_1\| + (1-\lambda) = \lambda' \|f'_1\| + (1-\lambda').$$

Subtracting we see that  $\lambda = \lambda'$ .

COROLLARY 5.2.9. Let  $K$  and  $K'$  be simplexes such that  $A_0(K)$  is linearly isometric to  $A_0(K')$ , then there is an affine homeomorphism of  $K$  onto  $K'$  which preserves the distinguished extreme point.

COROLLARY 5.2.10. Let  $\Sigma, \Sigma'$  be locally compact Hausdorff spaces such that  $C_0(\Sigma)$  is linearly isometric to  $C_0(\Sigma')$ , then  $\Sigma$  and  $\Sigma'$  are homeomorphic.

### 5.3. Direct sum decompositions.

If  $A$  is a vector space,  $A$  is termed a direct sum of subspaces  $A_1, A_2$  if each  $a \in A$  can be written uniquely as a sum  $a_1 + a_2$  with  $a_i \in A_i$ . If  $A$  is a Banach space, then the sum will be termed an  $M$ -direct sum if also  $\|a_1 + a_2\| = \max \{\|a_1\|, \|a_2\|\}$  whenever  $a_i \in A_i$ , and we then write  $A = A_1 \oplus_M A_2$ . Similarly, if  $\|a_1 + a_2\| = \|a_1\| + \|a_2\|$  whenever  $a_i \in A_i$ , then  $A$  is termed an  $L$ -direct sum, and we write  $A = A_1 \oplus_L A_2$ . We shall also abuse the language and say that  $A$  is a direct sum of  $A_1$  and  $A_2$  if  $A$  is a direct sum of subspaces that are linearly

isometric to each  $A_1$ . It is trivially seen that if  $A$  is either an  $L$ - or an  $M$ -direct sum of subspaces  $A_1$  and  $A_2$ , then  $A_1$  and  $A_2$  are closed. The following result is easily verified.

PROPOSITION 5.3.1. If  $A = A_1 \oplus_M A_2$  then  $A^* = A_1^* \oplus_L A_2^*$ , and if  $A = A_1 \oplus_L A_2$  then  $A^* = A_1^* \oplus_M A_2^*$ .

The result that we wish to generalise is one that was originally due to Eilenberg, [1]. He showed that if  $\Omega$  is a compact Hausdorff space and  $C(\Omega) = C_1 \oplus_M C_2$  then  $\Omega = \Omega_1 \cup \Omega_2$  with  $\Omega_1$  closed and  $C_1$  isometrically order isomorphic to  $C(\Omega_1)$ . This was extended to the space of continuous affine functions on a simplex by Jellet [2]. In this case, if  $A(K) = A_1 \oplus_M A_2$ , then there are complementary closed faces  $F_1, F_2$  of  $K$  such that  $A_1$  is linearly order isomorphic and isometric to  $A(F_1)$ . We will generalise this result to an arbitrary compact convex set, where the faces involved are split. In fact we can obtain results for the spaces  $A_0(C)$ .

We first prove the result in the reverse direction:

PROPOSITION 5.3.2. Let  $F_1, F_2$  be disjoint, complementary, closed split faces of a compact convex set  $K$ . Then  $A(K) = A(F_1) \oplus_M A(F_2)$ .

Proof. As the faces are split, any function that is affine on  $F_1$  and  $F_2$  can be extended to an affine function on  $K$ . By Lemma 5.2.3 if the function is continuous on  $F_1$  and  $F_2$ , then the extension is continuous. This shows that  $A(K) = A(F_1) \oplus A(F_2)$ . The relation between the norms arises because any continuous affine function on  $K$  attains its maximum modulus on  $\partial_e K$ , and hence on  $F_1 \cup F_2$ .

**PROPOSITION 5.3.3.** Let  $F_1, F_2$  be closed split faces of the cap  $C$ , such that  $F_1 \cap F_2 = \{0\}$  and  $\text{co}(F_1 \cup F_2) = C$ . Then we have  
 $A_0(C) = A_0(F_1) \oplus_M A_0(F_2)$ .

Proof. Suppose  $a_1 \in A_0(F_1)$  and  $a_2 \in A_0(F_2)$ . As  $F_1$  is a split face it has a complementary face,  $F_2 \cap \{k \in C : \|k\| = 1\}$ . There is an affine extension  $a$  of  $a_1$  and the restriction of  $a_2$  to this complementary face. As  $F_2$  is the convex hull of  $0$  and  $F_2 \cap \{k \in C : \|k\| = 1\}$ ,  $a|_{F_2} = a_2$ . Thus  $a$  extends  $a_1$  and  $a_2$ , so we may apply Lemma 5.2.3 to show that  $a$  is continuous.

Our ultimate aim is to prove the converse of these last two results, but first we prove a result on  $L$ -direct sum decompositions of base-normed spaces.

**THEOREM 5.3.4.** Let  $E$  be a base-normed Banach space, and let  $E = E_1 \oplus_L E_2$ . Then each  $E_i$  is an ideal in  $E$ .

Proof. Let  $x_1 \in E_1$ , and suppose that  $0 \leq y_1 + y_2 \leq x_1$ . (We shall assume that  $y_i \in E_i$ , etc.) Let  $z_1 + z_2 = x_1 - (y_1 + y_2) \geq 0$ . As the norm is additive on the positive cone of  $E$ , we have

$$\begin{aligned} \|x_1\| &= \|y_1 + y_2\| + \|z_1 + z_2\| \\ &= \|y_1\| + \|y_2\| + \|z_1\| + \|z_2\|. \end{aligned}$$

(the latter equality being because the sum is an  $L$ -direct sum).

Since the sum is direct,  $y_2 + z_2 = 0$ , so that  $x_1 = y_1 + z_1$ . Then we have:

$$\begin{aligned} \|y_1\| + \|z_1\| + 2\|y_2\| &= \|x_1\| \\ &= \|y_1 + z_1\| \\ &\leq \|y_1\| + \|z_1\|. \end{aligned}$$

Hence,  $y_2 = z_2 = 0$ , and thus  $E_1$  is an order ideal. Similarly

$E_2$  is an order ideal.

We must now show that each  $E_i$  is positively generated.

Let  $B = \{x \in E : x \geq 0, \|x\| = 1\}$ . We claim that  $E_1 \cap B$  and  $E_2 \cap B$  are complementary split faces of  $B$ . Let  $x \in B$ , then  $2x = x_1 + x_2$  with  $x_i \in E_i$ , and  $2 = 2\|x\| = \|x_1\| + \|x_2\|$ . If either  $x_i = 0$  then  $x$  belongs to  $E_1 \cap B$  or  $E_2 \cap B$ . If neither term is zero, then we can write

$$x = (\|x_1\|/2)(x_1/\|x_1\|) + (\|x_2\|/2)(x_2/\|x_2\|),$$

with  $\|x_1/\|x_1\|\| = 1$  and  $(\|x_1\|/2) + (\|x_2\|/2) = 1$ . As  $B$  is a face of the unit ball of  $E$ ,  $x_1/\|x_1\| \in B$ . It is now clear that the faces are complementary. That the faces are split follows from the fact that the sum is direct. If  $S_1 = \text{co}((E_1 \cap B) \cup -(E_1 \cap B))$ , then it is readily seen that  $S_1$  generates all of  $E_1$ , for if this were not so,  $S = \text{co}(S_1 \cup S_2)$  would not generate all of  $E$ . But  $S$  contains the open unit ball of  $E$ , so generates the whole space. Hence each  $E_i$  is an ideal, as required.

COROLLARY 5.3.5. Let  $\Omega$  be a compact Hausdorff space, and  $\mu$  a regular Borel measure on  $\Omega$ . If  $L^1(\Omega, \mu) = L_1 \oplus L_2$ , then  $\Omega = \Omega_1 \cup \Omega_2$ , with  $\Omega_i$  Borel subsets of  $\Omega$ , in such a way that  $L_i$  is isometrically order isomorphic to  $L(\Omega_i, \mu|_{\Omega_i})$ .

Proof. As  $\Omega$  is compact,  $\mu$  is finite so that  $1_\Omega \in L^1(\Omega, \mu)$ . Suppose  $1_\Omega = e_1 + e_2$  with  $e_i \in L_i$ . Let  $\Omega_1$  be a Borel subset of  $\Omega$  such that  $e_1$  is non-zero almost everywhere on  $\Omega_1$ , and zero almost everywhere on  $\Omega_2 = \Omega \setminus \Omega_1$ . Then  $e_2$  is zero almost everywhere on  $\Omega_1$ , for else there is a set  $U$  of measure  $> 0$  such that  $|e_2| > \varepsilon$  on  $U$ , and  $|e_1| > \varepsilon$  on  $U$ , for some  $\varepsilon > 0$ . We then have  $|e_1|, |e_2| \geq \varepsilon \chi_U \geq -|e_1|, -|e_2|$ . As each  $L_i$  is an ideal,

$\chi_U \in L_1$ . Hence the measure of  $U$  is zero. Now clearly  $e_2$  is 1 almost everywhere on  $\Omega_2$ , and  $e_1$  is 1 almost everywhere on  $\Omega_2$ . As  $L_1$  is an ideal,  $L^1(\Omega_1, \mu|_{\Omega_1}) \subset L_1$ . But  $L^1(\Omega, \mu) = L^1(\Omega_1, \mu|_{\Omega_1}) \oplus_L L^1(\Omega_2, \mu|_{\Omega_2})$ , so that  $L_1 = L^1(\Omega_1, \mu|_{\Omega_1})$ .

We mention here the following related problem. Given an  $L^p(\Omega, \mu)$  space and a direct sum decomposition  $L_1 \oplus L_2$  such that the norm satisfies  $\|a_1 + a_2\|^p = \|a_1\|^p + \|a_2\|^p$ , when is there a decomposition of  $\Omega$  into  $\Omega_1, \Omega_2$  such that  $L_1$  may be identified with  $L^p(\Omega_1, \mu|_{\Omega_1})$ ? We also note that there is no more general form of direct sum for which such a result is likely, because of the results of Bohnenblust [1] and Nakano [3].

**THEOREM 5.3.6.** If  $A = A(K)$  or  $A_0(C)$ , and  $A = A_1 \oplus_M A_2$  then each  $A_i$  is an ideal in  $A$ .

In particular, if  $A = A(K)$ , there exist complementary closed split faces  $F_1$  and  $F_2$  of  $K$  such that  $A_i$  is isometrically order isomorphic to  $A(F_i)$ . Similarly, if  $A = A_0(C)$ , then there exist closed split faces  $F_1, F_2$  of  $C$  such that  $F_1 \cap F_2 = \{0\}$ ,  $\text{co}(F_1 \cup F_2) = C$  and  $A_i = A_0(F_i)$ .

**Proof.** In either case  $A^*$  is base-normed, and  $A^* = A_1^* \oplus_L A_2^*$ . Suppose  $a_1 \in A_1$ , and  $a_1 \not\geq 0$ . In this case there is an  $f \in A_1^*$  such that  $f(a_1) < 0$  (since the positive cone in  $A$  is closed). By the last Theorem,  $f = f_1 + f_2$  with  $f_i \in A_i^*$ , and each  $f_i \geq 0$ . We then have  $f_1(a_1) < 0$  whilst  $f_1 \geq 0$ . If  $a_1 + a_2 \geq 0$ , then we have:

$$f_1(a_1) + f_2(a_2) \geq 0 \quad (\forall f_1, f_2 \geq 0)$$

so that  $f_1(a_1) \geq 0$  whenever  $f_1 \geq 0$ , and  $f_2(a_2) \geq 0$  if  $f_2 \geq 0$ .

It follows that  $a_1, a_2 \geq 0$ .

Now if  $a_1 \geq b = b_1 + b_2 \geq 0$ , then  $b_1, b_2 \geq 0$ . Also, as  $(a_1 - b_1) - b_2 \geq 0$ ,  $-b_2 \geq 0$ , so that  $b_2 = 0$ . Hence  $A_1$  is an order ideal.

In particular we note that the relative ordering on  $A_1$  considered as a subspace of  $A^*$ , coincides with the ordering as the dual of  $A_1$ . Also each  $A_i^*$  is weak\*-closed, and the weak\*-topologies on  $A_i^*$  considered as the dual of  $A_i$ , or as a subspace of  $A_i^*$  coincide. In the two cases considered,  $A_{i+}^*$  has either a weak\*-compact base or cap, so that  $A_i$  can be identified with  $A(K \cap A_i^*)$  or  $A_0(C \cap A_i^*)$  as the case may be. From Theorem 5.3.4 and the remarks at the beginning of this paragraph, these sets have precisely the properties claimed in the statement of the theorem. It only remains to note that these order ideals are positively generated to complete the proof.

We can state as immediate corollaries the special cases of this result that were already known.

COROLLARY 5.3.7. If  $K$  is a simplex and  $A(K) = A_1 \oplus_M A_2$ , then there exist complementary closed faces  $F_1$  and  $F_2$  of  $K$  such that  $A_1$  is isometrically order isomorphic to  $A(F_1)$ .

COROLLARY 5.3.8. If  $\Omega$  is a compact Hausdorff space, and  $C(\Omega) = C_1 \oplus_M C_2$ , then there exist closed subsets  $\Omega_1$  and  $\Omega_2$  of  $\Omega$  such that  $C_1$  is isometrically order isomorphic to  $C(\Omega_1)$ .

We can also state the following corollary, which does not seem to have appeared in print before.

COROLLARY 5.3.9. If  $\Sigma$  is a locally compact Hausdorff space and  $C_0(\Sigma) = C_1 \oplus_M C_2$ , then there are closed subsets  $\Sigma_1$  and

$\Sigma_2$  of  $\Sigma$  such that  $\Sigma = \Sigma_1 \cup \Sigma_2$  and  $C_1$  is isometrically order isomorphic to  $C_0(\Sigma_1)$ .

The last result that we wish to present is to show that extreme positive operators from  $A(K)$  into  $C(\Omega)$  induce direct sum decompositions of  $C(\Omega)$  from those of  $A(K)$ . Firstly we need a lemma:

LEMMA 5.3.10. Let  $\Omega$  be a compact Hausdorff space,  $K$  a compact convex set, and  $T$  an extreme point of  $\mathcal{A}(A(K), C(\Omega))$ . If  $a \in \tilde{A}(K)$  and  $b \in A(K)$ , then  $T(ab) = (T_a)(T_b)$ .

Proof. Suppose firstly that  $\frac{1}{4}1_K \leq a \leq \frac{3}{4}1_K$ , so that also  $\frac{1}{4}1_\Omega \leq T_a \leq \frac{3}{4}1_\Omega$ . Define for each  $b \in A(K)$ ,

$$T_1(b) = (T_a)^{-1} T(ab),$$

$$T_2(b) = (4/3)(T_b) - (1/3)(T_1(b)).$$

It is clear that  $T_1, T_2 \in \mathcal{A}(A(K), C(\Omega))$ . Moreover  $T = \frac{1}{4}T_1 + \frac{3}{4}T_2$ , so that  $T_1 = T_2 = T$ . Hence  $T(b) = (T_a)^{-1} T(ab)$ , so that  $T(ab) = (T_a)(T_b)$  as claimed. In general the result follows from finding  $m, n \neq 0$  such that  $\frac{1}{4}1_K \leq (a - m1_K)/n \leq \frac{3}{4}1_K$ .

THEOREM 5.3.11. Let  $\Omega$  be a compact Hausdorff space,  $K$  a compact convex set, and  $T$  an extreme point of  $\mathcal{A}(A(K), C(\Omega))$ . If  $A(K) = A(F_1) \oplus_M A(F_2)$ , then  $C(\Omega)$  can be written as a direct sum  $C(\Omega_1) \oplus_M C(\Omega_2)$  in such a way that  $T(A(F_1)) \subset C(\Omega_1)$ . If  $T_1 = T|_{A(F_1)}$ , then either  $\Omega_1 = \emptyset$  or  $T_1$  is an extreme point of  $\mathcal{A}(A(F_1), C(\Omega_1))$ .

Conversely, if  $A(K) = A(F_1) \oplus_M A(F_2)$ ,  $C(\Omega) = C(\Omega_1) \oplus_M C(\Omega_2)$  ( $F_1 \neq \emptyset$ , and  $\Omega_1 \neq \emptyset$ ) and  $T_1$  is an extreme point of  $\mathcal{A}(A(F_1), C(\Omega_1))$ , then the formula

$$(T_1 \oplus T_2)(a_1, a_2) = T_1 a_1 + T_2 a_2 \quad (1)$$

defines an extreme point of  $\mathcal{A}(A(K), C(\Omega))$ .

Proof. Let  $e_1$  denote the function that is identically one on  $F_1$  and zero on  $F_2$ , which exists by Proposition 5.3.2, and let  $e_2$  be defined similarly. It is clear that  $e_i \in \tilde{A}(K)$ . By Lemma 5.3.10 we see that  $T(e_1) = T(e_1^2) = (Te_1)^2$ . Thus  $(Te_1)(\omega) = 0$  or  $1$  for all  $\omega \in \Omega$ . As  $(T_1)(\omega) = (Te_1)(\omega) + (Te_2)(\omega) = 1$ , the sets  $\Omega_1 = \{\omega : (Te_1)(\omega) = 1\}$  are complementary subsets of  $\Omega$ , and these are clearly closed. If  $a_1 \in A(F_1)$ , then there is  $\lambda > 0$  such that  $\lambda e_1 \geq a_1 \geq -\lambda e_1$ . We thence see that  $\lambda(Te_1) \geq Ta_1 \geq -\lambda(Te_1)$ , and it follows that  $Ta_1|_{\Omega_2} = Ta_2|_{\Omega_1} = 0$ . I.e.  $Ta_1 \in C(\Omega_1)$ , so that  $T(A(F_1)) \subset C(\Omega_1)$ .

It is clear that if  $T_1$  is not identically zero, then  $T_1$  is a point of  $\mathcal{A}(A(F_1), C(\Omega_1))$ . Suppose  $T_2$  is not extreme, so that  $T_2 = \frac{1}{2}(V_2 + W_2)$ , where  $V_2, W_2 \in \mathcal{A}(A(F_2), C(\Omega_2))$ . The operators  $T_1 \oplus V_2$  and  $T_1 \oplus W_2$ , defined by (1), belong to  $\mathcal{A}(A(K), C(\Omega))$ . As  $T = T_1 \oplus T_2$  is extreme, it follows immediately that  $V_2 = W_2 = T_2$ , so that  $T_2$  is extreme. Similarly  $T_1$  is extreme.

For the converse, it is clear that  $T_1 \oplus T_2 \in \mathcal{A}(A(K), C(\Omega))$ . Suppose  $T_1 \oplus T_2$  is not extreme, so that  $2(T_1 \oplus T_2) = V + W$ , where  $V, W \in \mathcal{A}(A(K), C(\Omega))$ . Suppose  $a_1 \in A_1$  and  $a_1 \geq 0$ . Then we have:

$$2(T_1 a_1) = Va_1 + Wa_1$$

and if  $\omega \in \Omega_2$  then

$$0 = 2(T_1 a_1)(\omega) = Va_1(\omega) + Wa_1(\omega).$$

Since  $V, W \geq 0$  it follows that

$$Va_1(\omega) = Wa_1(\omega) = 0 \quad \text{whenever } \omega \in \Omega_2.$$

I.e.  $Va_1, Wa_1 \in C(\Omega_1)$ . As the positive cone of each  $A_i$  is generating, it follows that  $V(A(F_1)) \subset C(\Omega_1)$  and also that

$W(A(F_i)) \subset C(\Omega_i)$ . If  $V_i = V|_{A(F_i)}$  etc., then we have:

$$2T_i = V_i + W_i.$$

But  $V_i$  and  $W_i$  are points of  $\mathcal{A}(A(F_i), C(\Omega_i))$ , and the  $T_i$  are extreme, so that

$$T_i = V_i = W_i.$$

It is now immediate that

$$V = W = T_1 \oplus T_2,$$

so that  $T_1 \oplus T_2$  is extreme.

That this result cannot be generalised by allowing  $C(\Omega)$  to be replaced by  $A(K')$  for an arbitrary compact convex set, is easily seen. Indeed, let  $K$  be the unit interval in the  $X$ -axis in  $\mathbb{R}^2$ , and  $K'$  the unit square in  $\mathbb{R}^2$ . The extension operator, assigning to each  $a \in A(K)$  the function  $Ta : (x, y) \mapsto a(x)$ , is easily seen to be extreme. However the splitting of  $K$  between its two endpoints does not induce an  $M$ -direct sum decomposition of  $A(K')$ , although it does for  $T(A(K))$ . We would conjecture that  $C(\Omega)$  could be replaced by  $A(S)$ , for  $S$  a compact simplex, but have no proof of this.

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